REPORT 957

THE CALCULATION OF DOWNWASH BEHIND SUPERSONIC WINGS WITH AN APPLICATION TO TRIANGULAR PLAN FORMS

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SUMMARY

A method is developed consistent with the assumptions of small perturbation theory which provides a means of determining the downwash behind a wing in supersonic flow for a known load distribution. The analysis is based upon the use of supersonic doublets which are distributed over the plan form and wake of the wing in a manner determined from the wing loading.

The equivalence in subsonic and supersonic flow of the downwash at infinity corresponding to a given load distribution is proved. In order to introduce the manipulative techniques which are subsequently employed, the unswept wing of infinite span is treated for supersonic speeds. The principal application in this report, however, is concerned with the downwash behind a triangular wing with leading edges swept back of the Mach cone from the vertex. Complete solutions are given for the chord plane in the extended vortex wake of the wing and for the vertical plane of symmetry. An approximate solution is also provided for points in the vicinity of the center line of the wake.

INTRODUCTION

The linearization of the partial differential equation satisfied by the velocity potential for compressible flow yields, for subsonic flight speeds, an elliptic-type equation which is reducible by means of an elementary transformation to the basic equation in incompressible flow. As a consequence of this result, wing theory in the subsonic realm employs the same concepts and types of analyses that belong to classical incompressible theory. At supersonic speeds, the differential equation for the velocity potential is hyperbolic in type and for wing theory is equivalent to the two-dimensional wave equation of physics. In spite of the different character of the basic differential equation in the two flight regimes, certain formal equivalencies can be set up which are intuitively useful in the solutions of specific problems. In particular, the velocity potentials of a three-dimensional source and of a doublet each have analogous forms in the two cases. The solution of different boundary-value problems encountered in wing theory has been discussed in reference 1, and it has been shown how suitable distributions of sources and doublets may be used to determine the flow potential associated with a given lifting or nonlifting wing.

The calculation of downwash behind a wing, for incompressible flow, relies almost exclusively on the use of Prandtl's lifting-line theory which is, in turn, developed from the concept of a single horseshoe vortex. The conventional approach to the general downwash problem is to determine, first, the induced field of the simple horseshoe vortex by means of the Biot-Savart law and, then, from a knowledge of the span-wise distribution of loading over the wing, to calculate finally the induced field produced by a vortex sheet composed of superimposed vortices of varying span.

When downwash calculations are to be extended to the case of supersonic wings, it appears at first that the use of vortex sheets is inadmissible since no practical equivalent to the Biot-Savart law exists. It is, in fact, true that the horseshoe vortex no longer plays the outstanding role it has at low speeds. However, when a more detailed investigation is made of the underlying analysis, it becomes apparent that vortex theory and the Biot-Savart law can be developed from the initial use of a constant distribution of doublets over a given surface (e. g., see references 2 and 3). These doublets produce a discontinuity in the velocity potential at the surface, and, for incompressible theory, the curve which bounds the surface can be identified with a vortex curve possessing circulation. The proof of the Biot-Savart law and the introduction of vortex sheets are direct consequences of these basic ideas.

Since, as was shown in reference 1, supersonic boundary-value problems involving sources, sinks, and doublets can be solved in a manner analogous to that used in low-speed theory, a method is therefore provided for an attack on the downwash problem for supersonic plan forms through the use of doublet distributions. By means of this method the downwash immediately back of the trailing edge and at an infinite distance behind a wing will be derived and shown to agree with the previously published results of P. A. Lagerstrom (reference 4).

The present report has three principal aims: First, to outline the theoretical approach to the determination of the velocity potential of the flow field associated with a supersonic lifting surface and the subsequent calculation of the downwash; second, to apply the theory to the case of a triangular wing swept back of the Mach cone and to present the results of the complete calculations over the chord plane in the extended vortex wake of the wing and on the vertical plane of symmetry up to about 40 percent of a semispan; and, third, to serve as a guide through some of the more difficult mathematical manipulations so that the calculations can be extended to other plan forms. A simple first approximation is also advanced for the downwash variation about the axis of symmetry, and these approximate values of downwash are compared with those obtained from the exact calculations.

In the theoretical portion of the report, the boundary-value problem will be introduced and the solutions, obtained from Green's theorem, will be given for low-speed and supersonic flow. In the section of the report devoted to applications, the theory will be used to evaluate the potential function at an infinite distance downstream from a lifting wing. The theory is next applied to the case of the unswept wing of infinite span since the mathematical problems involved correspond closely to those for the more general case. From this application, a general procedure is developed for treating wings with supersonic trailing edges. The final application of the report will be devoted to the triangular wing. In all of these applications, it will be seen that the analytic expressions which have been obtained in supersonic theory for the load distributions over certain plan forms afford a means whereby the chordwise distribution of pressure may be introduced into the analysis, and, therefore, such expedients as lifting-line theory are no longer so essential.

The entire theory is postulated on the assumptions of thin-airfoil or small-perturbation theory and, consequently, thickness effects and lifting-plate solutions are additive. For the results that are given in the plane of the airfoil, the thickness effect, which is necessarily symmetrical with respect to this plane, is zero.

The material given in the present report is a combination of two previously published NACA Technical Notes (references 5 and 6).

LIST OF IMPORTANT SYMBOLS

a_0	velocity of sound in the free stream
b	span of wing
c_0	root chord of wing
E, E_0	complete elliptic integral of the second kind with modulus k, k_0, respectively $\left(E=\int_0^1\sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt\right)$
$E(t, k)$	incomplete elliptic integral of the second kind with argument t and modulus k $\left[E(t, k)=\int_0^t\sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt\right]$
H	$\frac{2\alpha V_0}{E_0\beta}$
k_0	$\sqrt{1-\theta_0^2}$
K	complete elliptic integral of the first kind with modulus k $\left[K=\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\right]$
$F(t, k)$	incomplete elliptic integral of the first kind with argument t and modulus k $\left[F(t, k)=\int_0^t\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\right]$
M_0	free-stream Mach number $\left(\frac{V_0}{a_0}\right)$
p	static pressure
Δp	p_l-p_u
q	free-stream dynamic pressure $\left(\frac{1}{2}\rho_0V_0^2\right)$
r	$\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$
r_c	$\sqrt{(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2(z-z_1)^2}$
u, v, w	perturbation velocity components in the direction of the x, y, and z axes, respectively
Δu_s	u_u-u_l
V_0	free-stream velocity
w_P	z component of velocity induced by doublet distribution over plan form
w_W	z component of velocity induced by doublet distribution over wake
w_0	$-V_0\alpha$
x, y, z	Cartesian coordinates of an arbitrary point
x_1, y_1, z_1	Cartesian coordinates of source or doublet position
x_0	$\frac{x}{c_0}$
y_0	$\frac{\beta y}{c_0}$
z_0	$\frac{\beta z}{c_0}$
α	angle of attack
β	$\sqrt{M_0^2-1}$
θ_0	$\beta\tan\psi$
μ_0	Mach angle $\left(\text{arc sin}\,\frac{1}{M_0}\right)$
ρ_0	density in free stream
Φ	perturbation velocity potential
$\Delta\Phi_s$	$\Phi_u-\Phi_l$
ψ	semivertex angle of triangular wing
———	sign denoting finite part of integral

SUBSCRIPTS

u	conditions on upper portion of surface
l	conditions on lower portion of surface
$L.E.$	conditions at leading edge
$T.E.$	conitions at trailing edge
W	wake
P	plan form
s	conditions on discontinuity surface (at $z_1=0$)
I, II, III	conditions in regions I, II and III on plan form (fig. 4)
A, B, C, D, E	conditions in regions A, B, C, D, or E, in wake of triangular wing (figs. 1 and 2)

THEORY

BOUNDARY CONDITIONS

The proposed problem is one of finding the downwash behind a flat plate which supports a loading consistent with its angle of attack and plan form. It will be assumed throughout the analysis that this load distribution is known. Such values were given for several plan forms in reference 7 and further results can be found in the literature on supersonic wings.

The load distribution over the wing may be obtained from a knowledge of the differences in pressures acting on the lower and upper surfaces. Moreover, in thin-airfoil theory, where boundary conditions are given in the $z=0$ plane (i. e., the plane of the wing), a simple relation istsex between local-pressure coefficient and the streamwise component of the perturbation velocity. Thus, assuming that the free-stream direction coincides with the positive x axis (fig. 1 (a)), and denoting by u the x component of the perturbation velocity, it follows that

$$\frac{\Delta p}{q}=\frac{p_l-p_u}{q}=\frac{2}{V_0}\ (u_u-u_l)=\frac{2\Delta u_s}{V_0} \tag{1}$$

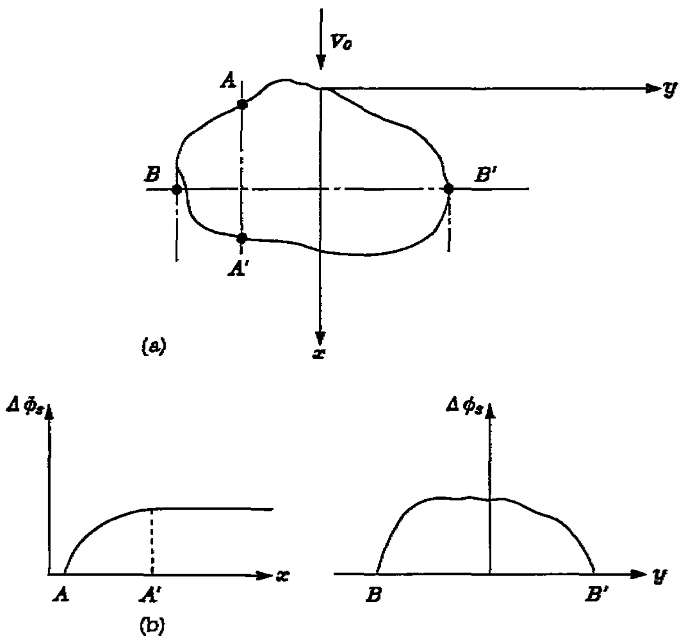
where the variables are defined in the table of the symbols. Furthermore, from the definition of the perturbation velocity potential Φ

$$\Phi=\int_a^x udx \tag{2}$$

where a is a point in a region at which the potential is zero. Combining equations (1) and (2), the jump in potential in the plane $z=0$ can be determined by integrating the jump in the u induced velocity or, what amounts to the same thing, the jump in load coefficient. Thus,

$$\Delta\Phi_s=\int_{L.E.}^x \Delta u_s dx_1=\frac{V_0}{2}\int_{L.E.}^x \left(\frac{\Delta p}{q}\right)dx_1 \tag{3}$$

where the integration extends from the leading edge to the point x and $\Delta\Phi_s$ represents the jump in Φ in the xy plane. Since load coefficiet $\Delta p/q$ must be zero off the wing and since u is an odd function in z, the value of u must be zero for all points off the wing in the xy plane. It follows that $\Delta\Phi_s$ remains constant at a given span station for all values of x beyond the trailing-edge position.



(a) Plan form.
(b) Sections showing distribution of $\Delta\Phi_s$.

FIGURE 1.—Sketch showing arbitrary lifting surface together with distribution of $\Delta\Phi_s$, the jump in perturbation velocity potential in the plane of the surface.

Figure 1 indicates an arbitrary lifting surface in the $z=0$ plane together with the distribution of $\Delta\Phi_s$ for given constant values of y and x. In both subsonic and supersonic theory, the wing together with the semi-infinite strip extending downstream of the wing form a discontinuity surface for the velocity potential, while $\Delta\Phi_s$ is equal to 0 throughout the remaining portion of the xy plane. These conditions, together with the fact that the vertical induced velocity w is a continuous function at $z=0$, are sufficient to determine Φ throughout space. The values of u, v, and w can then be found from the corresponding partial derivatives of Φ with respect to x, y, and z. The attention in the present report is centered on w, the downwash function.

SOLUTION TO BOUNDARY-VALUE PROBLEM

In reference 1, the solutions for boundary-value problems of the type under consideration were given for both incompressible and supersonic theory. The basic differential equations satisfied by the perturbation velocity potential are, for the two cases, respectively,

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \tag{4}$$

and

$$\beta^2\frac{\partial^2\Phi}{\partial x^2}-\frac{\partial^2\Phi}{\partial y^2}-\frac{\partial^2\Phi}{\partial z^2}=0 \tag{5}$$

Incompressible theory.—For boundary conditions prescribed in the $z=0$ plane; the solution of equation (4) is

$$\Phi(x,y,z)=-\frac{1}{4\pi}\int_\tau\int\left[\frac{1}{r_s}\left(\frac{\partial\Delta\Phi}{\partial z_1}\right)_s-(\Delta\Phi)_s\left(\frac{\partial}{\partial z_1}\frac{1}{r}\right)_s\right]dx_1dy_1 \tag{6}$$

where

$$r=\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$$

and τ is the area for which the integrand does not vanish. The terms $\frac{1}{4\pi r_s}$ and $\frac{1}{4\pi}\left(\frac{\partial}{\partial z_1}\frac{1}{r}\right)_s$ are equal to the velocity potential at x,y,z of a unit source and doublet situated at the point $x_1,y_1,0$. The remaining terms in the integrand, which determine the distribution of source and doublet strengths, must be found from known boundary conditions. If a lifting surface fixes the boundary conditions, induced vertical velocities on the upper and lower faces of the surface are equal so that

$$\frac{\partial\Phi_u}{\partial z_1}=\frac{\partial\Phi_l}{\partial z_1}$$

and

$$\Phi(x,y,z)=\frac{1}{4\pi}\int_\tau\int\Delta\Phi_s\left(\frac{\partial}{\partial z_1}\frac{1}{r}\right)_s dx_1dy_1 \tag{7}$$

Equations (6) and (7) are well known in potential theory (reference 3, p. 60), but the derivation usually employs the assumption that the value of Φ is zero at all points infinitely distant from the wing. This assumption cannot, of course, be made in aerodynamic applications where the discontinuity surface τ extends to $x=\infty$, as in the case of a lifting wing or lifting line with trailing vortices. These latter problems, with which this report is directly concerned, are of such a nature, however, that the induced effects at an infinite distance are confined to the plane $x=\infty$. An investigation of the derivation of equation (6) reveals that the conditions imposed on Φ, in general, can be relaxed sufficiently to permit a discontinuity in a strip of finite width along the entire extent of the x axis. The mathematical details of the derivation will not be given here but a statement of the restrictions on Φ at infinity is worthwhile. Thus, denoting by $\partial\Phi/\partial n$ the directional derivative of Φ taken normal to a prescribed surface, the following conditions apply:

1. The functions Φ and $\partial\Phi/\partial n$ are zero at all points having radius vectors which make finite (nonzero) angles with the positive x axis, the points lying on a spherical surface of infinite radius with center at the wing. (This preserves the usual potential theory assumptions except over the portion of the spherical surface forming the plane $x=\infty$.)
2. The values Φ and $\partial\Phi/\partial x$ are bounded at all points infinitely distant from the lifting surface and at a noninfinite distance from the positive x axis. (This condition places restrictions on the values of Φ and $\partial\Phi/\partial x$ in the plane $x=\infty$.)

Conditions (1) and (2) are satisfied for a lifting surface of finite span, and equation (7) is consequently applicable directly to the determination of the velocity potential. As an application of the equation, suppose a sheet of horseshoe vortices is situated as in figure 2 with bound vortices placed

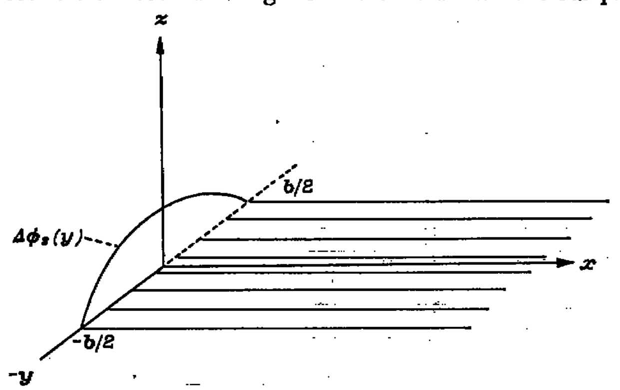


FIGURE 2.—Vortex sheet with bound vortices on y axis and distribution of circulation equal to $\Delta\Phi_s$.

on the y axis, trailing vortices extending parallel to the positive x axis, and has a spanwise distribution of circulation $\Delta\Phi$ symmetrical to the xz plane and defined for $-\frac{b}{2}\leq y\leq\frac{b}{2}$. Then the velocity potential corresponding to this vortex sheet is given by the expression

$$\Phi(x,y,z)=\frac{z}{4\pi}\int_{-b/2}^{b/2}\Delta\Phi_s\,dy_1\int_0^{\infty}\frac{dx_1}{[(x-x_1)^2+(y-y_1)^2+z^2]^{3/2}} \tag{8}$$

When $\Delta\Phi_s=$constant, a single horseshoe vortex results.

Supersonic theory.—For supersonic boundary-value problems associated with plan forms as indicated in figure 1 (a), where the known conditions are given in the $z=0$ plane, the general solution of equation (5) is given in reference 1 in the form

$$\Phi(x,y,z)=-\frac{1}{2\pi}\overline{\left|\int_\tau\int\left[\left(\frac{1}{r_c}\right)_s\left(\frac{\partial\Delta\Phi}{\partial z_1}\right)_s-(\Delta\Phi)_s\left(\frac{\partial}{\partial z_1}\frac{1}{r_c}\right)_s\right]dx_1dy_1\right.} \tag{9}$$

where

$$r_c=\sqrt{(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2(z-z_1)^2}$$

and the subscript s on the parentheses indicates that the function is to be evaluated at $z_1=0$. The region τ is that portion of the x_1y_1 plane bounded by the leading edge of the wing, the lines parallel to the x axis and stemming from the lateral tips of the wing, and the trace in the $z_1=0$ plane of the Mach forecone with vertex at the point x, y, z. The sign $\overline{|\quad}$ is to be read "finite part of" and was introduced by Hadamard (reference 8) as a manipulative technique with the property that

$$\overline{\left|\int_a^{x_0}\frac{A(x)dx}{(x_0-x)^{3/2}}\right.}=\int_a^{x_0}\frac{A(x)-A(x_0)}{(x_0-x)^{3/2}}dx-\frac{2A(x_0)}{(x_0-a)^{1/2}} \tag{10}$$

For purposes of calculations, this was modified in reference 1 to

$$\overline{\left|\int_a^{x_0}\frac{A(x)dx}{(x_0-x)^{3/2}}\right.}=\int_a^{*}\frac{A(x)dx}{(x_0-x)^{3/2}}=-F(a)-C \tag{11}$$

the asterisk indicating that no upper limit is to be substituted into the indefinite integral, the latter being determined as

$$F(x)+C$$

where

$$C=\lim_{x\to x_0}\left[\frac{2A(x_0)}{\sqrt{x_0-x}}-F(x)\right]$$

Equation (9) is the direct analogue of equation (6). The terms $\frac{1}{2\pi}\left(\frac{1}{r_c}\right)_s$ and $\frac{1}{2\pi}\left(\frac{\partial}{\partial z_1}\frac{1}{r_c}\right)_s$ are equal to the velocity potential at x, y, z of a unit supersonic source and doublet situated at the point x_1, y_1, 0, while the remaining terms in the integrand determine the distribution of source and doublet strength and are determined by the known boundary conditions.

When the potential function associated with a lifting surface is to be evaluated,

$$\frac{\partial\Phi_u}{\partial z_1}=\frac{\partial\Phi_l}{\partial z_1}$$

and equation (9) reduces to the form

$$\Phi(x,y,z)=\frac{1}{2\pi}\overline{\left|\int_\tau\int\Delta\Phi_s\left(\frac{\partial}{\partial z_1}\frac{1}{r_c}\right)_s dx_1dy_1\right.} \tag{12}$$

In application, the region of integration in equations (7) and (12) can be divided into areas occupied, respectively, by the plan form and the wake region. Thus, for equation (12),

$$\Phi(x,y,z)=\frac{-z\beta^2}{2\pi}\overline{\left|\int\int_{\text{plan form}}\frac{\Delta\Phi_s dx_1dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}\right.}-\frac{z\beta^2}{2\pi}\overline{\left|\int\int_{\text{wake}}\frac{\Delta\Phi_s dx_1dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}\right.} \tag{13}$$

Equation (13) presents a formal solution for the calculation of velocity potential and, subsequently, downwash for a given surface in terms of $\Delta\Phi_s$. Since $\Delta\Phi_s$ was related directly to load distribution in equation (3), it is apparent that the various known solutions to lifting-surface problems are directly applicable. The fact that supersonic theory permits the determination of load distribution in closed analytic form for many simple plan forms provides a distinct advantage that is lacking in subsonic theory wherein virtually all known results are available only in numerical form. Thus, theoretical analysis of problems involving supersonic flight speeds can be carried further before recourse to numerical methods is necessary.

APPLICATIONS

VALUE OF POTENTIAL FUNCTION AT $x=\infty$

It is possible to show, from equations (7) and (12), that the potential functions corresponding to a wing with fixed load distribution are identical at $x=\infty$ for incompressible and supersonic flow. Assuming $\Delta\Phi_s$ known, the values of (x, y, z) for the two cases are given, respectively, by the equations

$$\Phi(x,y,z)=\frac{z}{4\pi}\int\int_{\text{plan form}}\frac{\Delta\Phi_s dx_1 dy_1}{[(x-x_1)^2+(y-y_1)^2+z^2]^{3/2}}+\frac{z}{4\pi}\int\int_{\text{wake}}\frac{\Delta\Phi_s dx_1 dy_1}{[(x-x_1)^2+(y-y_1)^2+z^2]^{3/2}}$$

and

$$\Phi(x,y,z)=\frac{-\beta^2 z}{2\pi}\overline{\int\int_{\text{plan form}}\frac{\Delta\Phi_s dx_1 dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}}-\frac{\beta^2 z}{2\pi}\overline{\int\int_{\text{wake}}\frac{\Delta\Phi_s dx_1 dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}}$$

Since, however, $\Delta\Phi_s$ is finite, it follows immediately that for fixed values of y and z the integrals over the plan form in both equations approach zero as x increases indefinitely. Thus, denoting by $x_{T.E.}$ the value of x_1 at the trailing edge of the wing, the potential functions at $x=\infty$ are given by the expressions

$$\Phi(\infty,y,z)=\lim_{x\to\infty}\frac{z}{4\pi}\int_{-b/2}^{b/2}\Delta\Phi_s(x_{T.E.},y_1)dy_1\int_{x_{T.E.}}^{\infty}\frac{dx_1}{[(x-x_1)^2+(y-y_1)^2+z^2]^{3/2}}$$

and

$$\Phi(\infty,y,z)=\lim_{x\to\infty}\frac{-z\beta^2}{2\pi}\int_{-b/2}^{b/2}\Delta\Phi_s(x_{T.E.},y_1)dy_1\int_{x_{T.E.}}^{*}\frac{dx_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}$$

These relations can be integrated once to give for the subsonic case

$$\Phi(\infty,y,z)=\lim_{x\to\infty}\frac{-z}{4\pi}\int_{-b/2}^{b/2}\Delta\Phi_s(x_{T.E.},y_1)dy_1\left\{\frac{(x-x_1)}{[(y-y_1)^2+z^2]\sqrt{(x-x_1)^2+(y-y_1)^2+z^2}}\right\}_{x_{T.E.}}^{\infty}=\frac{z}{2\pi}\int_{-b/2}^{b/2}\frac{\Delta\Phi_s(x_{T.E.},y_1)dy_1}{(y-y_1)^2+z^2}\tag{14a}$$

and for the supersonic case

$$\Phi(\infty,y,z)=\lim_{x\to\infty}\frac{-z}{2\pi}\int_{-b/2}^{b/2}\Delta\Phi_s(x_{T.E.},y_1)dy_1\left\{\frac{(x-x_1)}{[(y-y_1)^2+z^2]\sqrt{(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2}}\right\}_{x_{T.E.}}^{*}=\frac{z}{2\pi}\int_{-b/2}^{b/2}\frac{\Delta\Phi_s(x_{T.E.},y_1)dy_1}{(y-y_1)^2+z^2}\tag{14b}$$

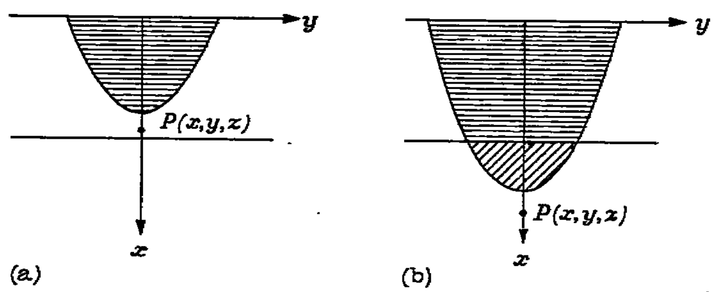
From these equations, it follows that the sidewash and downwash at $x=\infty$ are invariant with Mach number, provided the load distribution is fixed. In fact, their values depend solely on the spanwise load distribution, since the terms corresponding to the chordwise distribution disappeared in the analysis. This has been pointed out elsewhere in the literature. It should be stressed, however, that the result which has been obtained here states that equal span load distributions in the two cases yield equal values of the potential function at $x=\infty$. This does not imply that a wing at low and supersonic speeds maintains the same potential function at infinity. When the wing is kept fixed, the distribution of w on the wing is fixed, but the load distribution is a function of speed.

DOWNWASH ON AND OFF THE WING

As a further application, the unswept wing of infinite span will be treated for supersonic speeds. In this case, as is well known, the induced velocities are zero at all points downstream of the upper and lower Mach waves stemming from the trailing edge. An abrupt jump in vertical velocity therefore occurs at the trailing edge of the wing. Consideration of this jump for the unswept wing furnishes considerable insight into the nature of the mathematical difficulties inherent in the calculation of downwash on and off wings of arbitrary plan form. The calculation for the particular case will therefore be followed by a more general discussion which will be of value in connection with the later treatment of the triangular wing.

Unswept wing of infinite span.—The pressure distribution for the wing of infinite aspect ratio is constant. For this so-called Ackeret-type loading, $\Delta p/q$ is equal to $4\alpha/\beta$, so that, when the leading edge lies along the y axis

$$\Delta\Phi_s=\frac{V_0}{2}\int_0^{x_1}\frac{\Delta p}{q}\,dx=\frac{2\alpha V_0}{\beta}\,x_1\tag{15a}$$



(a) Point P ahead of trailing edge. (b) Point P behind trailing edge.

FIGURE 3.—Areas of integration for infinite span wing.

where α is angle of attack. In the wake

$$\Delta\Phi_s=\frac{V_0}{2}\int_0^{c_0}\frac{\Delta p}{q}\,dx=\frac{2\alpha V_0}{\beta}\,c_0\tag{15b}$$

The downwash, or vertical velocity, will first be found when the point is between the Mach waves from the leading and trailing edges and then when the point is downstream of the trailing-edge wave.

Since the wing is of infinite width and experiences no variation with y, it is possible to consider the problem at $y=0$. Thus, from equation (13), for the case when the point under consideration is between the Mach waves from the leading and trailing edges with its forecone cutting the wing as shown in figure 3(a),

$$\Phi=\lim_{\epsilon\to 0}\frac{-z\beta^2V_0\alpha}{\pi\beta}\int_0^{x-\beta\sqrt{\epsilon^2+z^2}} x_1\,dx_1 \left|\overline{2\int_\epsilon^{\frac{1}{\beta}\sqrt{(x-x_1)^2-\beta^2z^2}}\frac{dy_1}{[(x-x_1)^2-\beta^2y_1^2-\beta^2z^2]^{3/2}}}\right.$$

where the symmetry of the problem with respect to the y_1 axis has been used. Computing the finite part of the integral

$$\Phi=\lim_{\epsilon\to 0}\frac{2z\beta V_0\alpha}{\pi}\epsilon$$

$$\int_0^{x-\beta\sqrt{\epsilon^2+z^2}}\frac{x_1\,dx_1}{[(x-x_1)^2-\beta^2z^2]\sqrt{(x-x_1)^2-\beta^2\epsilon^2-\beta^2z^2}} \quad (16)$$

and

$$w=\frac{\partial\Phi}{\partial z}=\frac{\partial}{\partial z}\lim_{\epsilon\to 0}\frac{2z\beta V_0\alpha\epsilon}{\pi}$$

$$\int_0^{x-\beta\sqrt{\epsilon^2+z^2}}\frac{x_1\,dx_1}{[(x-x_1)^2-\beta^2z^2]\sqrt{(x-x_1)^2-\beta^2\epsilon^2-\beta^2z^2}} \quad (17)$$

Equations (16) and (17) can be evaluated directly to give the results

$$\left.\begin{aligned}\Phi&=\frac{V_0\alpha}{\beta}\left(x\frac{z}{|z|}-\beta z\right)\\ w&=-V_0\alpha\end{aligned}\right\} \quad (18)$$

For a point behind the trailing-edge wave (fig. 3 (b)), the two quantities can be determined in a similar manner. Thus,

$$\Phi=\lim_{\epsilon\to 0}\frac{2z\beta V_0\alpha\epsilon}{\pi}\int_0^{c_0}\frac{x_1\,dx_1}{[(x-x_1)^2-\beta^2z^2]\sqrt{(x-x_1)^2-\beta^2z^2-\beta^2\epsilon^2}}-\frac{2z\beta V_0\alpha c_0}{\pi}\int_0^{\frac{1}{\beta}\sqrt{(x-c_0)^2-\beta^2z^2}}dy_1\left|\overline{\int_{c_0}^{x-\beta\sqrt{y_1^2+z^2}}\frac{dx_1}{[(x-x_1)^2-\beta^2y_1^2-\beta^2z^2]^{3/2}}}\right. \quad (19)$$

and w is given by the partial derivative of Φ with respect to z. The term containing the single integral is zero, since the integral itself is bounded for all values of ϵ, while the term containing the double integral is readily calculable. Thus, the values of the velocity potential and the downwash for a point behind the trailing-edge wave are given by the relations

$$\left.\begin{aligned}\Phi&=\frac{V_0\alpha c_0}{\beta}\frac{z}{|z|}\\ w&=0\end{aligned}\right\} \quad (20)$$

These results are the familiar equations associated with two-dimensional supersonic flatplate theory.

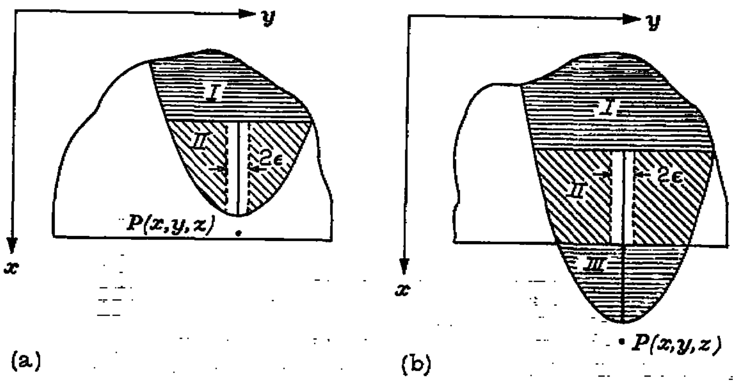
The point of principal interest in this development is the jump in the induced vertical velocity w in passing from a point just ahead of the trailing-edge wave to a point immediately behind the wave. A study of equations (17) and (19) shows that this jump is the result of the discontinuity in the contribution to the downwash of the term containing the single integral. Ahead of the trailing-edge wave, this term yielded the result that

$$w=-V_0\alpha$$

whereas behind the wave, the contribution of the term to w was zero.

The method of attack used in the study of the unswept wing can be generalized to apply to arbitrary plan forms. A discussion of this case follows.

Arbitrary plan forms.—As will be shown later, for any plan form with supersonic trailing edge, the jump in the value of w in the plane of the wing at the trailing edge can be calculated directly by means of simple momentum methods. At this point, however, it is of more interest to consider in a general manner the nature of the integrations involved when the point x, y, z is either ahead of or behind the trailing-edge wave. Figures 4 (a) and 4 (b) show a plan form with a straight trailing edge with areas of integration indicated for the point P in each of the two positions. (The straight trailing edge is not a necessary restriction and is only introduced for convenience of notation.) The regions of integration are divided under the assumption that the first integration on the plan form in equation (13) will be made with respect to y_1. When the point P is ahead of the trailing-edge wave, therefore, the contribution of the wake is zero and the integration over the plan form is made to conform with regions I and II. When the point is behind the trailing-edge wave, three integrals are evaluated corresponding to regions I, II, and III. In the case of the infinite aspect ratio, unswept wing region I was, of course, nonexistent and, in general, no essential difficulty in regard to the limits of integration is introduced by this region regardless of where P is situated. In region II, however, the problem must be treated in more detail.



(a) Point P ahead of trailing edge. (b) Point P behind trailing edge.

FIGURE 4.—Areas of integration for arbitrary plan form with supersonic trailing edge.

Consider first the case when P is ahead of the wave and denote by Φ_{IIa} the contribution of one side of region II to the total potential. Then

$$\Phi_{IIa}=\lim_{\epsilon\to 0}\frac{-\beta^2 z}{2\pi}\int_x^{x-\beta\sqrt{\epsilon^2+z^2}}dx_1\left|\int_{y+\epsilon}^{Y_1}\frac{\Delta\Phi_1(x_1,y_1)dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}\right. \tag{21a}$$

where

$$Y_1=y+\frac{1}{\beta}\sqrt{(x-x_1)^2-\beta^2z^2}$$

Similarly, when P is behind the wave and the same subscript notation is used to refer to one side of region II, the value of Φ_{IIa} is

$$\Phi_{IIa}=\lim_{\epsilon\to 0}\frac{-\beta^2 z}{2\pi}\int_x^{c_0}dx_1\left|\int_{y+\epsilon}^{Y_1}\frac{\Delta\Phi_1(x_1,y_1)dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}\right. \tag{21b}$$

where Y_1 is as defined above.

The contributions of the other side of region II to the potential will not be considered separately as the behavior is identical. When P lies ahead of the wave, ϵ appears in the limits for integration with respect to both y_1 and x_1. This corresponds to the situation in equations (16) and (17) and, as for that problem, the limiting process is carried out after the integration is completed. When P is behind the wave, it is not necessary to defer the limiting process, since

$$\Phi_{IIa}=\frac{-\beta^2 z}{2\pi}\int_x^{c_0}dx_1\lim_{\epsilon\to 0}\left|\int_{y+\epsilon}^{Y_1}\frac{\Delta\Phi_1(x_1,y_1)dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}\right.$$

and if $\{\ \}$ represents the integrand, then

$$\left|\int_{y+\epsilon}^{Y_1}\{\ \}dy_1=\right|\int_y^{Y_1}\{\ \}dy_1-\int_y^{y+\epsilon}\{\ \}dy_1$$

But since

$$\frac{\Delta\Phi_1(x_1,y_1)}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}\leq M$$

for $c_0\leq x_1\leq x_0$ (i. e., $\{\ \}$ is bounded for all values of x_1 in the interval of the first integration); and further, since

$$\lim_{\epsilon\to 0}M\int_y^{y+\epsilon}dy_1=\lim_{\epsilon\to 0}M\epsilon=0$$

therefore, for P situated behind the trailing-edge wave, the contribution of region IIa is given by

$$\Phi_{IIa}=\frac{-\beta^2 z}{2\pi}\int_x^{c_0}dx_1\left|\int_y^{Y_1}\frac{\Delta\Phi_1(x_1,y_1)dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}\right.\ \cdots \tag{22}$$

The significance of this result is that, when the point P at x, y, z is behind the Mach wave from a supersonic trailing edge, the limiting process associated with region II need not be considered. When P is ahead of the Mach wave, the term ϵ must be retained in the analysis and the limiting process used. As was previously noted, the general analysis developed in this report places no restriction on the orientation of the trailing edge; however, it should be pointed out that region II exists only for the case in which the trailing edge is supersonic. Therefore, the jump in downwash, obtained from the integration over region II is associated only with supersonic trailing edges; whereas both the downwash and loading are continuous across a subsonic trailing edge.

TRIANGULAR WING

Consider a triangular wing (fig. 5) with leading edges swept

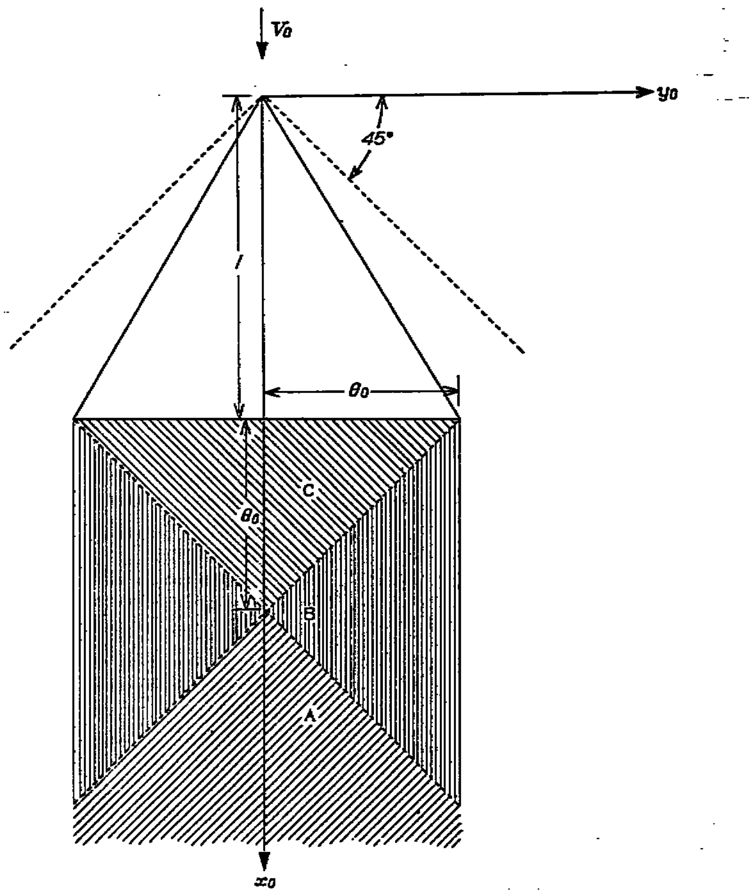


FIGURE 5.—Regions A, B, and C for triangular wing in x_0y_0 plane.

back of the Mach cone from the vertex. The loading over the wing is known to be (references 7 and 9)

$$\frac{\Delta p}{q}=\frac{4\theta_0{}^2\alpha x_1}{E_0\beta\sqrt{\theta_0{}^2x_1{}^2-\beta^2y_1{}^2}} \tag{23}$$

where E_0 is the complete elliptic integral of the second kind with modulus $k_0=\sqrt{1-\theta_0{}^2}$ and $\theta_0=\beta\tan\psi$, ψ being the semi-vertex angle of the triangle. From equation (3)

$$\Delta\Phi_s=H\sqrt{\theta_0{}^2x_1{}^2-\beta^2y_1{}^2} \tag{24}$$

where

$$H=\frac{2\alpha V_0}{E_0\beta} \tag{25}$$

Setting, for convenience, $\Phi=\Phi_P+\Phi_W$ the velocity potential at x, y, z is given by equation (13) to be the sum of the two expressions

$$\Phi_P=-\frac{zH\beta^2}{2\pi}\left|\int\int_{\text{plan form}}\frac{\sqrt{\theta_0{}^2x_1{}^2-\beta^2y_1{}^2}\,dx_1\,dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}\right. \tag{26}$$

$$\Phi_W=-\frac{zH\beta^2}{2\pi}\left|\int\int_{\text{wake}}\frac{\sqrt{\theta_0{}^2x_1{}^2-\beta^2y_1{}^2}\,dx_1\,dy_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2z^2]^{3/2}}\right. \tag{27}$$

Equation (26) represents the contribution to the velocity potential furnished by the doublets distributed over the plan form while equation (27) represents the contribution furnished by doublets in the wake. The latter equation is the mathematical equivalent in supersonic flow of the subsonic velocity potential of a sheet of horseshoe vortices corresponding to an elliptic span load distribution. Equations (14a) and (14b) showed that the expression for Φ_W at $x=\infty$ is identical to the velocity potential of the subsonic vortex sheet. However, in the vicinity of the $x=c_0$ line, the behavior is entirely different.

In the present report, equations (26) and (27) will be applied to the determination of downwash in the xy and xz planes.

SOLUTION IN THE XY PLANE

Effect of doublets in the plan form.—For the purpose of integration, it is convenient to divide the area behind the wing into three regions as shown in figure 5. The division lines separating these regions are formed by the Mach cone traces from the trailing-edge tips.

The following symbols will be used in the derivation of the expressions for downwash in the xy plane induced by the distribution of doublets over the plan form of a triangular wing swept behind the Mach cone:

E_1, E_2, E_3	complete elliptic integrals of the second kind with moduli k_1, k_2, and k_3, respectively
K_1, K_2, K_3	complete elliptic integrals of the first kind with moduli k_1, k_2, and k_3, respectively
k_1	$\sqrt{\frac{\delta_1}{\gamma_1}\left(\frac{\mu'-\gamma_1}{\delta_1-\mu'}\right)}$
k_2	$\left(\frac{\gamma_2-\mu}{\mu-\delta_2}\right)\left(\frac{\delta_2-\mu'}{\gamma_2-\mu'}\right)$
k_3	$\sqrt{\frac{\gamma_3}{\delta_3}\left(\frac{\mu'-\delta_3}{\mu'-\gamma_3}\right)}$
γ_1	$\frac{(\mu\mu'+\xi^2)-\sqrt{(\xi^2-\mu^2)(\xi^2-\mu'^2)}}{\mu+\mu'}$
γ_2	$\frac{\xi(\mu+\mu')-\sqrt{2\xi(\mu'-\mu)(\mu+\xi)(\mu'-\xi)}}{\mu-\mu'+2\xi}$
γ_3	$\frac{(\mu\mu'+\xi^2)+\sqrt{(\xi^2-\mu^2)(\xi^2-\mu'^2)}}{\mu+\mu'}$
δ_1	$\frac{(\mu\mu'+\xi^2)+\sqrt{(\xi^2-\mu^2)(\xi^2-\mu'^2)}}{\mu+\mu'}$
δ_2	$\frac{\xi(\mu+\mu')+\sqrt{2\xi(\mu'-\mu)(\mu+\xi)(\mu'-\xi)}}{\mu-\mu'+2\xi}$
δ_3	$\frac{(\mu\mu'+\xi^2)-\sqrt{(\xi^2-\mu^2)(\xi^2-\mu'^2)}}{\mu+\mu'}$
η	$\frac{\beta(y-y_1)}{c_0}$
μ	$y_0-\frac{\theta_0 x_1}{c_0}$
μ'	$y_0+\frac{\theta_0 x_1}{c_0}$
ξ	$\frac{x-x_1}{c_0}$
ξ_0	x_0-1

The downwash $w_P(x,y,0)$ may be obtained by considering $\lim_{z\to 0}\frac{\partial\Phi_P}{\partial z}$ in equation (26). It can be shown that, in this case, this limiting process corresponds to taking the partial derivative of equation (26) with respect to z and then simply setting z equal to zero. Thus the expressions for w_P in the regions A, B, and C are, respectively,

$$w_{P_A}=-\frac{H\beta}{2\pi}\left|\int_{\xi_0}^{x_0}d\xi\int_{\mu}^{\mu'}\frac{\sqrt{(\mu'-\eta)(\eta-\mu)}}{(\xi^2-\eta^2)^{3/2}}d\eta=-\frac{H\beta}{2\pi}\right|\int_{\xi_0}^{x_0}I_1\,d\xi \tag{28}$$

$$w_{P_B}=-\frac{H\beta}{2\pi}\left[\left|\int_{\frac{y_0+\theta_0x_0}{1+\theta_0}}^{x_0}d\xi\int_{\mu}^{\mu'}\frac{\sqrt{(\mu'-\eta)(\eta-\mu)}}{(\xi^2-\eta^2)^{3/2}}d\eta+\right.\right.$$
$$\left.\left.\int_{\xi_0}^{\frac{y_0+\theta_0x_0}{1+\theta_0}}d\xi\int_{\mu}^{\xi}\frac{\sqrt{(\mu'-\eta)(\eta-\mu)}}{(\xi^2-\eta^2)^{3/2}}d\eta\right|\right]$$
$$=-\frac{H\beta}{2\pi}\left(\left|\int_{\frac{y_0+\theta_0x_0}{1+\theta_0}}^{x_0}I_1d\xi+\right|\int_{\xi_0}^{\frac{y_0+\theta_0x_0}{1+\theta_0}}I_2d\xi\right) \tag{29}$$

$$w_{P_C}=-\frac{H\beta}{2\pi}\left[\left|\int_{\frac{y_0+\theta_0x_0}{1+\theta_0}}^{x_0}d\xi\int_{\mu}^{\mu'}\frac{\sqrt{(\mu'-\eta)(\eta-\mu)}}{(\xi^2-\eta^2)^{3/2}}d\eta+\right.\right.$$
$$\int_{\frac{-y_0+\theta_0x_0}{1+\theta_0}}^{\frac{y_0+\theta_0x_0}{1+\theta_0}}d\xi\int_{\mu}^{\xi}\frac{\sqrt{(\mu'-\eta)(\eta-\mu)}}{(\xi^2-\eta^2)^{3/2}}d\eta+$$
$$\left.\left.\int_{\xi_0}^{\frac{-y_0+\theta_0x_0}{1+\theta_0}}d\xi\int_{-\xi}^{\xi}\frac{\sqrt{(\mu'-\eta)(\eta-\mu)}}{(\xi^2-\eta^2)^{3/2}}d\eta\right|\right]$$
$$=-\frac{H\beta}{2\pi}\left(\left|\int_{\frac{y_0+\theta_0x_0}{1+\theta_0}}^{x_0}I_1d\xi+\right|\int_{\frac{-y_0+\theta_0x_0}{1+\theta_0}}^{\frac{y_0+\theta_0x_0}{1+\theta_0}}I_2d\xi+\left|\int_{\xi_0}^{\frac{-y_0+\theta_0x_0}{1+\theta_0}}I_3d\xi\right.\right) \tag{30}$$

The solution of the three integrals I_1, I_2, and I_3 will be discussed in Appendix A.

The expressions for downwash in regions A, B, and C may then be expressed as the following single integrals which can be handled by standard numerical methods:

$$w_{P_A}=-\frac{H\beta}{\pi}\int_{\xi_0}^{x_0}\frac{1}{2\xi^2}\sqrt{\frac{2\xi(\mu'-\mu)}{k_1}}\,(K_1-E_1)d\xi \tag{31}$$

$$w_{P_B}=-\frac{H\beta}{\pi}\left[\int_{\frac{y_0+\theta_0x_0}{1+\theta_0}}^{x_0}\frac{1}{2\xi^2}\sqrt{\frac{2\xi(\mu'-\mu)}{k_1}}\,(K_1-E_1)d\xi+\right.$$
$$\int_{\xi_0}^{\frac{y_0+\theta_0x_0}{1+\theta_0}}\left\{K_2\sqrt{\frac{k_2}{(\xi-\mu)(\xi+\mu')}}\left[\frac{\delta_2(\mu-\mu'+2\xi)-2\xi\mu)}{\xi^2}\right]-\right.$$
$$\left.\left.\frac{E_2}{2\xi^2}\sqrt{\frac{(\xi-\mu)(\xi+\mu')}{k_2}}\right\}d\xi\right] \tag{32}$$

$$w_{P_C}=-\frac{H\beta}{\pi}\left[\int_{\frac{y_0+\theta_0x_0}{1+\theta_0}}^{x_0}\frac{1}{2\xi^2}\sqrt{\frac{2\xi(\mu'-\mu)}{k_1}}\ (K_1-E_1)d\xi+\right.$$

$$\int_{\frac{-y_0+\theta_0x_0}{1+\theta_0}}^{\frac{y_0+\theta_0x_0}{1+\theta_0}}\left\{K_2\sqrt{\frac{k_2}{(\xi-\mu)(\xi+\mu')}}\left[\frac{\delta_2(\mu-\mu'+2\xi)-2\xi\mu}{\xi^2}\right]-\right.$$

$$\left.\frac{E_2}{2\xi^2}\sqrt{\frac{(\xi-\mu)(\xi+\mu')}{k_2}}\right\}d\xi+$$

$$\left.\int_{\xi_0}^{\frac{-y_0+\theta_0x_0}{1+\theta_0}}\frac{1}{2\xi^2}\sqrt{\frac{2\xi(\mu'-\mu)}{k_3}}\ (K_3-E_3)d\xi\right]$$

Effect of doublets in the wake.—The study of the downwash induced on the xy plane by the doublets distributed over the wake will also be divided into the three regions indicated in figure 5, and the symbols listed as follows will be used in the derivations:

a_A	$\frac{\delta_A}{\gamma_A}\left(\frac{\gamma_A-\nu'}{\delta_A-\nu'}\right)$
a_B	$\frac{\delta_B}{\gamma_B}\left(\frac{\gamma_B-\nu'}{\delta_B-\nu'}\right)$
a_C	$\frac{\gamma_C}{\delta_C}\left(\frac{\nu'-\delta_C}{\gamma_C-\nu'}\right)$
E_A, E_B, E_C	complete elliptic integrals of the second kind with moduli k_A, k_B, and k_C, respectively
$E\left(\frac{1}{a_A}, k_A\right)$, $E\left(\frac{1}{a_B}, k_B\right)$, $E\left(\frac{a_C}{k_C}, k_C\right)$	incomplete elliptic integrals of the second kind with arguments $1/a_A$, $1/a_B$, and a_C/k_C, and with moduli k_A, k_B, and k_C, respectively
K_A, K_B, K_C	complete elliptic integrals of the first kind with moduli k_A, k_B, and k_C, respectively
$F\left(\frac{1}{a_A}, k_A\right)$, $F\left(\frac{1}{a_B}, k_B\right)$, $F\left(\frac{a_C}{k_C}, k_C\right)$	incomplete elliptic integrals of the first kind with arguments $1/a_A$, $1/a_B$, and a_C/k_C, and moduli k_A, k_B, and k_C, respectively
k_A	$\sqrt{\frac{\delta_A}{\gamma_A}\left(\frac{\gamma_A-\nu'}{\delta_A-\nu'}\right)}$
k_B	$\left(\frac{\delta_B-\nu}{\nu-\gamma_B}\right)\left(\frac{\gamma_B-\nu'}{\delta_B-\nu'}\right)$
k_C	$\sqrt{\frac{\gamma_C}{\delta_C}\left(\frac{\nu'-\delta_C}{\gamma_C-\nu'}\right)}$
L_1L_2	undefined limits of integration
x_0	$\frac{x}{c_0}$
y_0	$\frac{\beta y}{c_0}$
z_0	$\frac{\beta z}{c_0}$
γ_A	$\frac{(\nu\nu'+\xi_0^2)-\sqrt{(\xi_0^2-\nu^2)(\xi_0^2-\nu'^2)}}{\nu+\nu'}$
γ_B	$\frac{\xi_0(\nu+\nu')-\sqrt{2\xi_0(\nu-\nu')(\nu-\xi_0)(\xi_0+\nu')}}{\nu'-\nu+2\xi_0}$
γ_C	$\frac{(\nu\nu'+\xi_0^2)+\sqrt{(\xi_0^2-\nu^2)(\xi_0^2-\nu'^2)}}{\nu+\nu'}$
δ_A	$\frac{(\nu\nu'+\xi_0^2)+\sqrt{(\xi_0^2-\nu^2)(\xi_0^2-\nu'^2)}}{\nu+\nu'}$
δ_B	$\frac{\xi_0(\nu+\nu')+\sqrt{2\xi_0(\nu-\nu')(\nu-\xi_0)(\xi_0+\nu')}}{\nu'-\nu+2\xi_0}$
δ_C	$\frac{(\nu\nu'+\xi_0^2)-\sqrt{(\xi_0^2-\nu^2)(\xi_0^2-\nu'^2)}}{\nu+\nu'}$
η	$\frac{\beta(y-y_1)}{c_0}$
ν	$y_0+\theta_0$
ν'	$y_0-\theta_0$
ξ_0	x_0-1

On integrating equation (27) with respect to x_1 and using the notation just presented

$$\Phi_W=\frac{H\beta}{2\pi}\,\frac{\xi_0z_0c_0}{\beta}\int_{L_2}^{L_1}\frac{\sqrt{(\nu-\eta)(\eta-\nu')}}{(\eta^2+z_0^2)\sqrt{\xi_0^2-\eta^2-z_0^2}}\,d\eta \tag{34}$$

The limits on the integral as previously noted differ in the regions A, B, and C shown in figure 5; however, in each case the limits are roots of one of the two radicals in the integrand.

It is desirable to express equation (34) in a different form in order to obtain an expression for downwash in the plane of the airfoil. Integrating by parts

$$\Phi_W=\frac{H\beta}{2\pi}\,\frac{c_0}{\beta}\left\{\left[\sqrt{(\nu-\eta)(\eta-\nu')}\ \tan^{-1}\frac{\xi_0\eta}{z_0\sqrt{\xi_0^2-\eta^2-z_0^2}}\right]_{L_2}^{L_1}-\right.$$

$$\left.\int_{L_2}^{L_1}\frac{\nu+\nu'-2\eta}{2\sqrt{(\nu-\eta)(\eta-\nu')}}\tan^{-1}\frac{\xi_0\eta}{z_0\sqrt{\xi_0^2-\eta^2-z_0^2}}\,d\eta\right\} \tag{35}$$

When $\lim_{z_0\to 0}\frac{\partial\Phi_W}{\partial z}$ is considered, it can be shown that in all three regions the contribution to the downwash made by the doublets in the wake is given by the expression

$$w_W=\frac{H\beta}{2\pi\xi_0}\int_{L_2}^{L_1}\frac{\nu+\nu'-2\eta}{2\eta\sqrt{(\nu-\eta)(\eta-\nu')}}\ \sqrt{\xi_0^2-\eta^2}\,d\eta$$

The solution of equation (36) in regions A, B, and C will be considered separately.

Region A

In region A, $L_1=\nu$ and $L_2=\nu'$. The substitution $\eta=\frac{\gamma_A+\delta_A t}{1+t}$ eliminates the linear term in the radical of the integrand, and equation (36) becomes

$$w_{W_A}=\frac{H\beta}{2\pi\xi_0}\sqrt{\frac{\xi_0^2-\gamma_A^2}{(\nu-\gamma_A)(\gamma_A-\nu')}}\,(\delta_A-\gamma_A)\int_{\frac{\nu'-\gamma_A}{\delta_A-\nu'}}^{\frac{\nu-\gamma_A}{\delta_A-\nu}}\left[\frac{\nu+\nu'}{2(\gamma_A+\delta_A t)(1+t)}\right]\left[-\frac{1}{(1+t)^2}\right]\sqrt{\frac{1-\left(\frac{\xi_0^2-\delta_A^2}{\xi_0^2-\gamma_A^2}\right)t^2}{1-\left(\frac{\delta_A-\nu}{\nu-\gamma_A}\right)^2t^2}}\,dt \tag{37}$$

The identities

$$\xi_0^2=\gamma_A\delta_A$$

and

$$(\delta_A-\nu)(\nu'-\gamma_A)+(\delta_A-\nu')(\nu-\gamma_A)=0$$

are useful in the integration.

The transformation $\omega=\frac{\delta_A-\nu}{\nu-\gamma_A}t$ is next introduced, and the expression for downwash becomes

$$w_{W_A}=\frac{H\beta}{2\pi}\sqrt{\frac{\xi_0^2-\gamma_A^2}{(\nu-\gamma_A)(\gamma_A-\nu')}}\,(\delta_A-\gamma_A)\left(\frac{\nu-\gamma_A}{\delta_A-\nu}\right)\left[\int_{-1}^{1}\frac{\nu+\nu'}{2\gamma_A(1+a_A\omega)\left(1+\frac{\gamma_A}{\delta_A}a_A\omega\right)}\sqrt{\frac{1-k_A^2\omega^2}{1-\omega^2}}d\omega-\int_{-1}^{1}\frac{1}{\left(1+\frac{\gamma_A}{\delta_A}a_A\omega\right)^2}\sqrt{\frac{1-k_A^2\omega^2}{1-\omega^2}}d\omega\right]$$

Integrating the second term by parts and applying the fundamental properties of even and odd functions yields

$$w_{W_A}=\frac{H\beta}{2\pi\xi_0}\sqrt{\frac{\xi_0^2-\gamma_A^2}{(\nu-\gamma_A)(\gamma_A-\nu')}}\,(\delta_A-\gamma_A)\left(\frac{\nu-\gamma_A}{\delta_A-\nu}\right)\left\{\int_{-1}^{1}-\frac{\nu+\nu'}{2\gamma_A}\left[1-\frac{1}{1-a_A^2\omega^2}-\frac{1}{1-\left(\frac{\gamma_A}{\delta_A}\right)^2a_A^2\omega^2}\right]\frac{d\omega}{\sqrt{(1-k_A^2\omega^2)(1-\omega^2)}}+\right.$$

$$\left.\left[-\frac{\delta_A}{\gamma_A a_A\left(1+\frac{\gamma_A}{\delta_A}a_A\omega\right)}\sqrt{\frac{1-k_A^2\omega^2}{1-\omega^2}}\right]_{-1}^{1}+\int_{-1}^{1}\frac{\delta_A^2(1-k_A^2)}{\delta_A^2-\gamma_A^2a_A^2}\left[\frac{\omega^2}{1-\omega^2}-\frac{1}{1-\left(\frac{\gamma_A a_A}{\delta_A}\right)^2\omega^2}+1\right]\frac{d\omega}{\sqrt{(1-k_A^2\omega^2)(1-\omega^2)}}\right\}=$$

$$\frac{H\beta}{2\pi\xi_0}\sqrt{\frac{\xi_0^2-\gamma_A^2}{(\nu-\gamma_A)(\gamma_A-\nu')}}\,(\delta_A-\gamma_A)\left(\frac{\nu-\gamma_A}{\delta_A-\nu}\right)\left\{\left[\frac{\delta_A}{\gamma_A a_A\left(1+\frac{\gamma_A}{\delta_A}a_A\omega\right)}\sqrt{\frac{1-k_A^2\omega^2}{1-\omega^2}}\right]_{-1}^{1}+\right.$$

$$\left.2\int_0^1\frac{\delta_A^2(1-k_A^2)}{(\delta_A^2-\gamma_A^2a_A^2)}\frac{\omega^2d\omega}{(1-\omega^2)^{3/2}\sqrt{1-k_A^2\omega^2}}+2\int_0^1\frac{\nu+\nu'}{2\gamma_A}\frac{d\omega}{(1-a_A^2\omega^2)\sqrt{(1-k_A^2\omega^2)(1-\omega^2)}}\right\} \tag{38}$$

The Jacobian transformation $\omega=sn\ u$ reduces the integrals in equation (38) to standard elliptic forms (reference 10), and

$$w_{W_A}=-\frac{H\beta}{\pi\xi_0}\left[(\xi_0^2-\nu^2)(\xi_0^2-\nu'^2)\right]^{1/4}\sqrt{1-k_A^2}\left\{\frac{a_A}{\sqrt{(k_A^2-a_A^2)(1-a_A^2)}}\left[K_AE\left(\frac{1}{a_A},k_A\right)-E_AF\left(\frac{1}{a_A},k_A\right)\right]+\frac{E_A}{1-k_A^2}\right\} \tag{39}$$

Region B

In region B, $L_1=\xi_0$ and $L_2=\nu'$ and equation (36) may be written

$$w_{W_B}=\frac{H\beta}{2\pi\xi_0}\int_{\nu'}^{\xi_0}\left[\left(\frac{\nu+\nu'}{2\eta}\right)\xi_0^2-\left(\frac{\nu+\nu'}{2}\right)\eta-\xi_0^2+\eta^2\right]\frac{d\eta}{\sqrt{[(\nu-\eta)(\xi_0+\eta)][(\xi_0-\eta)(\eta-\nu')]}} \tag{40}$$

The transformations $\eta=\frac{\gamma_B+\delta_B t}{1+t}$ and $\omega=\frac{\delta_B-\nu'}{\gamma_B-\nu'}t$ where

$$(\gamma_B-\nu')(\xi_0-\delta_B)+(\xi_0-\gamma_B)(\delta_B-\nu')=0$$

and

$$(\xi_0+\gamma_B)(\nu-\delta_B)+(\xi_0+\delta_B)(\nu-\gamma_B)=0$$

reduce equation (40) to

$$w_{W_B}=\frac{H\beta}{2\pi\xi_0}\left[\frac{\delta_B-\gamma_B}{\nu-\gamma_B}\sqrt{\frac{\nu-\delta_B}{(\delta_B-\nu')(\xi_0^2-\delta_B^2)}}\right]\int_{-1}^{1}\left\{\left[\frac{(\nu+\nu')\xi_0^2}{2\gamma_B(1+a_B\omega)}\left(1+\frac{\gamma_B}{\delta_B}a_B\omega\right)-\frac{(\nu+\nu')\gamma_B(1+a_B\omega)}{2\left(1+\frac{\gamma_B}{\delta_B}a_B\omega\right)}-\xi_0^2+\frac{\gamma_B^2(1+a_B\omega)^2}{\left(1+\frac{\gamma_B}{\delta_B}a_B\right)^2}\right]\frac{d\omega}{\sqrt{(1-k_B^2\omega^2)(1-\omega^2)}}\right\}$$

$$=\frac{H\beta}{2\pi\xi_0}\left[\frac{\delta_B-\gamma_B}{\nu-\gamma_B}\sqrt{\frac{\nu-\delta_B}{(\delta_B-\nu')(\xi_0^2-\delta_B^2)}}\right]\left\{\left[-\frac{\frac{\delta_B}{\gamma_B a_B}}{1+\frac{\gamma_B}{\delta_B}a_B\omega}\frac{1}{\sqrt{(1-k_B^2\omega^2)(1-\omega^2)}}\right]_{-1}^{1}+\int_{-1}^{1}\left[(\delta_B^2-\xi_0^2)\left(1-\frac{\nu+\nu'}{2\delta_B}\right)+\right.\right.$$

$$\left.\frac{\xi_0^2(\nu+\nu')(\delta_B-\gamma_B)}{2\gamma_B\delta_B(1-a_B^2\omega^2)}-\frac{(\delta_B-\gamma_B)^2}{\left(1-\frac{\gamma_B^2a_B^2}{\delta_B^2}\right)(1-\omega^2)}-\frac{(\delta_B-\gamma_B)^2}{\left(k_B^2-\frac{\gamma_B^2a_B^2}{\delta_B^2}\right)(1-k_B^2\omega^2)}\right]\frac{d\omega}{\sqrt{(1-k_B^2\omega^2)(1-\omega^2)}}\Bigg\}$$

When the transformation $\omega=sn\,u$ is made, equation (41) is readily integrable (reference 10), and, after algebraic simplification, may be written

$$w_{W_B}=-\frac{H\beta}{\pi\xi_0}\frac{2a_B(\delta_B-\nu')^2}{\delta_B^2(\nu'^2-1)}\frac{\sqrt{(\nu-1)(\nu'+1)}}{(1+a_B)}\left\{(1-k_B)K_B-\frac{a_BE_B}{(a_B-k_B)}-\frac{(1-k_B)(a_B^2+k_B)a_B\left[K_BE\left(\frac{1}{a_B},k_B\right)-E_BF\left(\frac{1}{a_B},k_B\right)\right]}{(a_B-k_B)\sqrt{a_B^2-1)(a_B^2-k_B^2)}}\right\} \tag{42}$$

Region C

In region C, $L_1=\xi_0$ and $L_2=-\xi_0$, and equation (36) is written

$$w_{W_C}=\frac{H\beta}{2\pi\xi_0}\int_{-\xi_0}^{\xi_0}\frac{\nu+\nu'-2\eta}{2\eta\sqrt{(\nu-\eta)\ (\eta-\nu')}}\sqrt{\xi_0^2-\eta^2}\,d\eta \tag{43}$$

The deriviation of the expression for w_{W_C} is similar to that for w_{W_A} with the exception that in this case the substitution $\omega=\sqrt{\frac{\delta_C}{\gamma_C}}\,t$ is made, and equation (43) may be written

$$w_{W_C}=-\frac{H\beta}{\pi\xi_0}[(\xi_0^2-\nu^2)\ (\xi_0^2-\nu'^2)]^{1/4}\sqrt{1-k_C^2}\left\{\frac{-a_C}{\sqrt{(k_C^2-a_C^2)\ (1-a_C^2)}}\left[\quad\left(\frac{a_C}{k_C},k_C\right)-E_CF\left(\frac{a_C}{k_C},k_C\right)\right]+\frac{E_C}{1-k_C^2}-K_C\right\} \tag{44}$$

SOLUTION IN THE XZ PLANE

Just as in the study of downwash in the xy plane, so also in its study in the xz plane it is convenient to consider separately the effects of the doublets distributed over the plan form and the wake. The subscript notation for w_W and w_P is the same as before and again w is equal to w_W+w_P.

Effect of doublets in the plan form.—In the xz plane two regions are indicated in figure 6. Region E lies between the

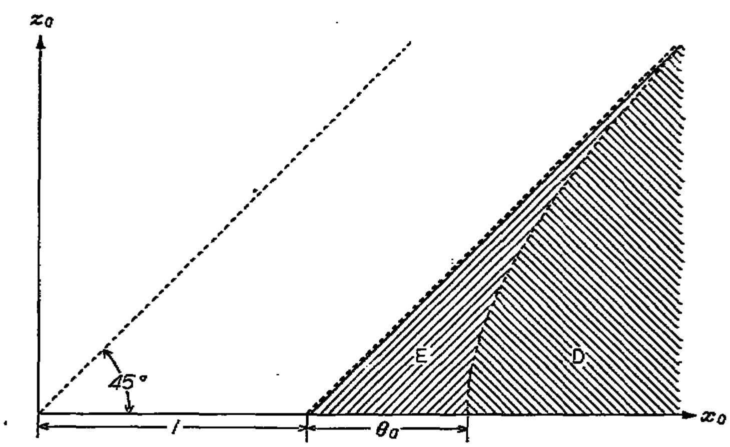


FIGURE 6.—Regions D and E in the x_0z_0 plane for a triangular wing.

Mach wedge from the trailing edge and the line of intersection of the two cones from the trailing-edge tips. Region D connects this region to infinity. Again the limits of integration form the basis of the division into the two regions.

The symbols listed below will be used in the derivation of the expressions for downwash in the xz plane, induced by the distribution of doublets over the plan form of the wing.

$E_4,E_5,E_{4_0}E_{5_0}$	complete elliptic integrals of the second kind with moduli k_4, k_5, k_{4_0}, and k_{5_0}, respectively
K_4,K_5,K_{4_0},K_{5_0}	complete elliptic integrals of the first kind with moduli k_4, k_5, k_{4_0}, and k_{5_0}, respectively
k_4	$\frac{\theta_0x_1}{\sqrt{(x-x_1)^2-\beta^2z^2}}$
k_{4_0}	$\frac{\theta_0c_0}{\sqrt{(x-c_0)^2-\beta^2z^2}}$
k_5	$\frac{\sqrt{(x-x_1)^2-\beta^2z^2}}{\theta_0x_1}$
k_{5_0}	$\frac{\sqrt{(x-c_0)^2-\beta^2z^2}}{\theta_0c_0}$

Region D

In region D, equation (26) is written

$$\Phi_{P_D}=-\frac{zH\beta^2}{\pi}\int^{c_0}dx_1\int_0^{\frac{\theta_0x_1}{\beta}}\frac{\sqrt{\theta_0^2x_1^2-\beta^2y_1^2}}{[(x-x_1)^2-\beta^2y_1^2-\beta^2z^2]^{3/2}}dy_1$$

Integrating with respect to y_1 leads to

$$\Phi_{P_D}=-\frac{zH\beta}{\pi}\int_0^{c_0}\frac{k_4}{\theta_0x_1}\,(K_4-E_4)dx_1$$

Changing the variable of integration gives

$$\Phi_{P_D}=-\frac{zH\beta}{\pi}\int_0^{k_{4_0}}\frac{\theta_0[xk_4-\sqrt{\theta_0^2x^2+\beta^2z^2(k_4^2-\theta_0^2)}]}{\sqrt{\theta_0^2x^2+\beta^2z^2(k_4^2-\theta_0^2)}}\left(\frac{K_4-E_4}{k_4^2-\theta_0^2}\right)dk_4$$

Taking the partial derivative of Φ_{P_D} with respect to z gives the expression for the downwash as

$$W_{P_D}=-\frac{H\beta}{\pi}\left\{\frac{k_{4_0}^2\beta^2z^2}{\theta_0^2c_0^2}\left(\frac{K_{4_0}-E_{4_0}}{k_{4_0}^2-\theta_0^2}\right)\frac{\theta_0}{\sqrt{\theta_0^2x^2+\beta^2z^2(k_{4_0}^2-\theta_0^2)}}\right.$$

$$[xk_{4_0}-\sqrt{\theta_0^2x^2+\beta^2z^2(k_{4_0}^2-\theta_0^2)}]+$$

$$\int_0^{k_{4_0}}\left(\frac{x^2k_4\theta_0^2}{[\theta_0^2x^2+\beta^2z^2(k_{4_0}^2-\theta_0^2)]^{3/2}}-1\right)\left(\frac{\theta_0}{k_4^2-\theta_0^2}\right)$$

$$\left.(K_4-E_4)dk_4\right\} \tag{45}$$

Region E

In region E, a similar derivation yields the solution

$$w_{P_E}=-\frac{H\beta}{\pi}\left[\frac{\beta^2z^2}{\theta_0^2c_0^2k_{5_0}^2}\left(\frac{K_{5_0}-E_{5_0}}{1-k_{5_0}^2\theta_0^2}\right)\right.$$

$$\frac{\theta_0\left[x-\sqrt{x^2k_{5_0}^2\theta_0^2+\beta^2z^2(1-k_{5_0}^2\theta_0^2)}\right]}{\sqrt{x^2k_{5_0}^2\theta_0^2+\beta^2z^2(1-k_{5_0}^2\theta_0^2)}}+$$

$$\int_{k_{5_0}}^{1}\frac{K_5-E_5}{k_5}\frac{\theta_0}{1-k_5^2\theta_0^2}\left\{\frac{x^3k_5^2\theta_0^2}{[x^2k_5^2\theta_0^2+\beta^2z^2(1-k_5^2\theta_0^2)]^{3/2}}-1\right\}dk_5+$$

$$\left.\int_0^1(K_4-E_4)\frac{\theta_0}{k_4^2-\theta_0^2}\left\{\frac{x^3k_4\theta_0^2}{[\theta_0^2x^2+\beta^2z^2(k_4^2-\theta_0^2)]^{3/2}}-1\right\}dk_4\right] \tag{46}$$

Effect of doublets in the wake.—The limits of integration again necessitate the division of the portion of the plane behind the trailing-edge wave into two regions, D and E. The following list of symbols will be used in the derivation of downwash in the xz plane, induced by the doublets distributed over the vortex wake:

a_D	$\dfrac{\sqrt{(x-c_0)^2-\beta^2z^2}}{\beta z}$
a'_D	$\dfrac{\sqrt{(x-c_0)^2-\beta^2z^2}}{x-c_0}$
a_E	$\dfrac{\theta_0c_0}{\beta z}$
a'_E	$\dfrac{\theta_0c_0}{\sqrt{\beta^2z^2+\theta_0^2c_0^2}}$
E_D, E_E	complete elliptic integrals of the second kind with moduli k_D and k_E, respectively
$E(a'_D,k'_D)$, $E(a'_E,k'_E)$	incomplete elliptic integrals of the second kind with arguments a'_D and a'_E and moduli k'_D and k'_E, respectively
K_D, K_E	complete elliptic integrals of the first kind with moduli k_D and k_E, respectively
$F(a'_D,k'_D)$, $F(a'_E,k'_E)$	incomplete elliptic integrals of the first kind with arguments a'_D and a'_E and moduli k'_D and k'_E, respectively
k_D	$\dfrac{\theta_0c_0}{\sqrt{(x-c_0)^2-\beta^2z^2}}$
k'_D	$\sqrt{1-k_D^2}$
k_E	$\dfrac{\sqrt{(x-c_0)^2-\beta^2z^2}}{\theta_0c_0}$
k'_E	$\sqrt{1-k_E^2}$
$sn^2(i\alpha_D)$	$-a_D^2$
$sn^2(i\alpha_E)$	$-a_E^2$

Region D

In region D, equation (27) becomes

$$\Phi_{W_D}=-\frac{zH\beta^2}{\pi}\left|\int_0^{\frac{\theta_0c_0}{\beta}}\sqrt{\theta_0^2c_0^2-\beta^2y_1^2}\,dy_1\int_{c_0}^{x-\beta\sqrt{y_1^2+z^2}}\frac{dx_1}{[(x-x_1)^2-\beta^2(y_1^2+z^2)]^{3/2}}\right. \tag{47}$$

Integrating with respect to x_1, and using the definition of ,

$$\Phi_{W_D}=\frac{zH(x-c_0)}{\pi}\int_0^{\frac{\theta_0c_0}{\beta}}\frac{\sqrt{\theta_0^2c_0^2-\beta^2y_1^2}}{(y_1^2+z^2)\sqrt{(x-c_0)^2-\beta^2y_1^2-\beta^2z^2}}dy_1$$

$$=\frac{H(x-c_0)}{\pi}\frac{\theta_0^2c_0^2}{\beta z\sqrt{(x-c_0)^2-\beta^2z^2}}\int^1\frac{\sqrt{1-\omega^2}}{\left(1+\frac{\theta_0^2c_0^2}{\beta^2z^2}\omega^2\right)\sqrt{1-k_D^2\omega^2}}d\omega$$

where

$$\omega=\frac{\beta y_1}{\theta_0c_0}$$

Applying the Jacobian transformation $\omega=sn\ u$

$$\Phi_{W_D}=\frac{H(x-c_0)\theta_0^2c_0^2}{\pi\beta z\sqrt{(x-c_0)^2-\beta^2z^2}}\int_0^{K_D}\frac{1-sn^2u}{1+a_D^2k_D^2sn^2u}du \tag{48}$$

The expression for Φ_{W_D} as given in equation (48) is integrable (reference 10) and becomes

$$\Phi_{W_D}=\frac{H(x-c_0)\theta_0^2c_0^2}{\pi\beta z\sqrt{(x-c_0)^2-\beta^2z^2}}\left\{K_D-\left(\frac{K_D\sqrt{1+a_D^2k_D^2}}{k_D^2\sqrt{1+a_D^2}}\right)\left(\frac{1}{ia_D}\right)\right.$$

$$\left[iF(a'_D,k'_D)+\frac{ia_D\sqrt{1+a_D^2k_D^2}}{\sqrt{1+a_D^2}}-\right.$$

$$\left.\left.iE(a'_D,k'_D)-iF(a'_D,k'_D)\frac{E_D}{K_D}\right]\right\}=-$$

$$\frac{H\beta z}{\pi}\left\{a'_Dk'^2_DK_D+\frac{\sqrt{1-a'^2_Dk'^2_D}}{\sqrt{1-a'^2_D}}\right.$$

$$\left.\left[K_DF(a'_D,k'_D)-E_DF(a'_D,k'_D)-K_DE(a'_D,k'_D)\right]\right\} \tag{49}$$

Since it may be shown that

$$\frac{\partial}{\partial z}[K_DF(a'_D,k'_D)-E_DF(a'_D,k'_D)-K_DE(a'_D,k'_D)]=-$$

$$\frac{\beta}{\sqrt{\beta^2z^2+\theta_0^2c_0^2}}\left[(K_D-E_D)a'_D-K_D\frac{(1-a'^2_Dk'^2_D)}{a'_D}\right]$$

the expression for the downwash w_{W_D} obtained by taking the partial derivative of Φ_{W_D} with respect to z may be written

$$w_{W_D}=\frac{H\beta}{\pi}\left\{-\frac{E_D}{a'_D}+\frac{K_D(1-a'^2_D)}{a'_D}-\frac{\sqrt{1-a'^2_D}}{\sqrt{1-a'^2_Dk'^2_D}}[K_DF(a'_D,k'_D)-\right.$$

$$\left.E_DF(a'_D,k'_D)-K_DE(a'_D,k'_D)]\right\} \tag{50}$$

Region E

In region E a similar derivation gives

$$w_{W_E}=\frac{H\beta}{\pi}\left\{\frac{\sqrt{1-a'^2_Ek'^2_E}}{a'_E}\left(\frac{K_E-E_E}{k_E^2}-a'^2_EK_E\right)-\right.$$
$$\left.\sqrt{1-a'^2_E}\,[K_EF(a'_E,k'_E)-E_EF(a'_E,k'_E)-K_EE(a'_E,k'_E)]\right\} \tag{51}$$

Conditions at the trailing edge.—The value of the vertical induced velocity immediately ahead of and behind the trailing-edge wave must, of course, be determinable directly from equations (21) and (22), respectively, by setting $x=c_0\pm z\beta$. If, however, the discussion is restricted to the $z=0$ plane, a much simpler method exists for finding the downwash at these points. The approach taken here follows essentially that given by Lagerstrom in reference 4.

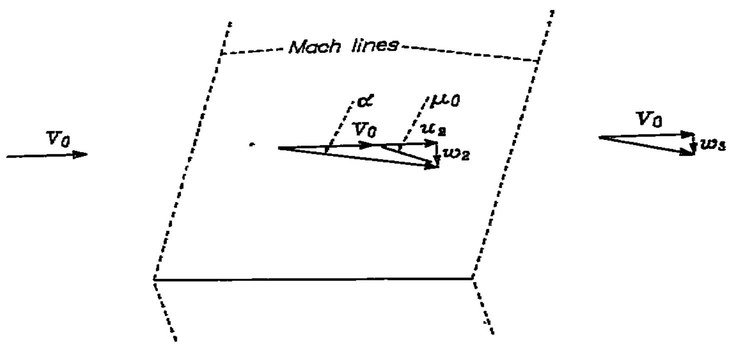


FIGURE 7.—Sketch of velocity vectors of the air before reaching, on, and after leaving supersonic airfoil.

Let conditions just ahead of the trailing-edge wave be denoted by the subscript 2 and conditions just behind the wave by the subscript 3. Figure 7 shows a section of a given wing in the plane y=constant. The Mach waves at the leading and trailing edge make the angle $\mu_0=\text{arc sin } 1/M_0$ with the $z=0$ plane, and the wing is presumed to be at angle of attack α. Assuming the trailing edge to be normal to the free-stream direction, the variation in the x component in velocity when passing through the trailing-edge wave can be treated as a two-dimensional problem with the condition imposed that $u_3=0$ in the $z=0$ plane.

It is known that continuity of flow together with balance of tangential momentum across the wave lead to the result that the component of velocity tangential to the wave is continuous. The tangential components of velocity, V_t, immediately ahead of and behind the wave are given, respectively, by the expressions

$$(V_t)_2=(V_0+u_2)\cos\mu_0+w_2\sin\mu_0$$
$$(V_t)_3=V_0\cos\mu_0+w_3\sin\mu_0$$

Equating these relations, it follows that

$$w_3=w_2+u_2\cot\mu_0$$
$$=-V_0\alpha+\frac{V_0\beta}{4}\left(\frac{\Delta p}{q}\right)_2$$

From equation (23)

$$\left(\frac{\Delta p}{q}\right)_2=\frac{4\theta_0^2\alpha c_0}{E_0\beta\sqrt{\theta_0^2c_0^2-\beta^2y^2}}$$

from which it follows that

$$w_3=-\frac{H\beta}{2}\left(E_0-\frac{\theta_0^2}{\sqrt{\theta_0^2-y_0^2}}\right) \tag{52}$$

Approximate values of downwash near center line of wake.—The values of downwash which were obtained on the xy and xz planes of the wake were exact solutions subjected to no restrictions other than those originally imposed by the use of the linearized equations of flow. On the center line equations (31), (33), (39), and (44) reduce to the considerably simpler expressions

$$(w_{P_A})_{y=0}=-\frac{H\beta}{\pi}\int_0^{\frac{\theta_0}{x_0-1}}\frac{K_1-E_1}{k_1+\theta_0}\,dk_1 \tag{53}$$

$$(w_{P_C})_{y=0}=-\frac{H\beta}{\pi}\left[\int_0^1\frac{K_1-E_1}{k_1+\theta_0}\,dk_1-\int_1^{\frac{x_0-1}{\theta_0}}\frac{K_3-E_3}{k_3^2(1+\theta_0k_3)}\,dk_3\right] \tag{54}$$

$$(w_{W_A})_{y=0}=-\frac{H\beta}{\pi}E_A \tag{55}$$

$$(w_{W_C})_{y=0}=-\frac{H\beta}{\pi}\,\frac{E_C-(1-k_C^2)K_C}{k_C} \tag{56}$$

where

$$(k_1)_{y=0}=\frac{\theta_0x_1}{x-x_1}$$

$$(k_3)_{y=0}=\frac{x-x_1}{\theta_0x_1}$$

$$(k_A)_{y=0}=\frac{\theta_0}{x_0-1}$$

$$(k_C)_{y=0}=\frac{x_0-1}{\theta_0}$$

Since the above expressions are relatively simple to compute, an approximate method based on the generalized Taylor's expansion in the vicinity of the line $y=z=0$ can be formulated which reduces the tediousness of the calculations and gives a good indication of the variation in the downwash function in a portion of the wake for points near the center line. The next higher terms in the expansion can be found without difficulty for the region bounded as follows:

(a) $-\frac{1}{2}b<y<\frac{1}{2}b$

(b) Both y and z lie within the Mach cones from the trailing-edge tips

The problem resolves itself into one of finding the first nonvanishing coefficients of y_0 and z_0 in the series

$$w/w_0=A_0+A_1z_0+B_1y_0+A_2z_0^2+C_2z_0y_0+B_2y_0^2+\ldots \quad (57)$$

where

$$w_0=-\frac{E_0}{2H\beta}$$

The value of A_0 is known from equations (53) and (55) to be

$$A_0=\frac{2}{\pi E_0}\left(E_A+\int_0^{\frac{\theta_0}{x_0-1}}\frac{K_1-E_1}{k_1+\theta_0}\,dk_1\right)$$

From equations (28) and (36) the expression for total downwash may be written

$$\frac{w\pi}{H\beta}=\frac{1}{2(x_0-1)}\int_{y_0-\theta_0}^{y_0+\theta_0}\frac{y_0-\eta}{\eta}\sqrt{\frac{(x_0-1)^2-\eta^2}{\theta_0^2-(y_0-\eta)^2}}\,d\eta-$$

$$\frac{1}{2}\int_{x_0-1}^{x_0}d\xi\int_{y_0-\theta_0(x_0-\xi)}^{y_0+\theta_0(x_0-\xi)}\frac{\sqrt{\theta_0^2(x_0-\xi)^2-(y_0-\eta)^2}}{(\xi^2-\eta^2)^{3/2}}\,d\eta$$

The coefficient B_1 in the expansion is given by $\left(\frac{\partial w}{\partial y_0}\right)_{y_0=0}$. Carrying out the differentiation, with proper regard for the singularity in the first integral, it follows that $B_1=0$. Similarly, it can be shown that $C_2=0$, while the coefficient $B_2=\frac{1}{2}\left(\frac{\partial^2 w}{\partial y_0^2}\right)_{y_0=0}$ is given by the expression

$$B_2=\frac{1}{\pi E_0\theta_0^2}\left(2K_A-\frac{2-k_A^2}{1-k_A^2}E_A\right)-$$

$$\frac{1}{\pi E_0\theta_0^2x_0^2}\int_0^{k_A}(k_1+\theta_0)\left[E_1\frac{1+k_1^2}{(1-k_1^2)^2}-K_1\frac{1}{1-k_1^2}\right]dk_1 \quad (58)$$

where the variables have previously been defined.

In order to calculate the variation with z, it is necessary to evaluate $A_1=\left(\frac{\partial^2\Phi}{\partial z_0^2}\right)_{z_0=0}$ where

$$\frac{\Phi\pi}{H\beta}=\beta\int_0^{\frac{\theta_0c_0}{\beta}}\frac{y_1}{\sqrt{\theta_0^2c_0^2-\beta^2y_1^2}}\arctan\frac{y_1(x-c_0)}{z\sqrt{(x-c_0)^2-\beta^2y_1^2-\beta^2z^2}}\,dy_1-$$

$$z\beta\int_0^{c_0}dx_1\int_0^{\frac{\theta_0x_1}{\beta}}\frac{\sqrt{\theta_0^2x_1^2-\beta^2y_1^2}}{[(x-x_1)^2-\beta^2y_1^2-\beta^2z^2]^{3/2}}\,dy_1$$

The double integral contributes nothing to the coefficient, and the remaining portion of the expression can be evaluated without integrating by differentiating twice and using Cauchy's integral theorem

$$\oint\frac{f(y_1)dy_1}{(y_1-iz)^2}=2\pi i\left(\frac{\partial f}{\partial y_1}\right)_{y_1=iz}$$

Thus

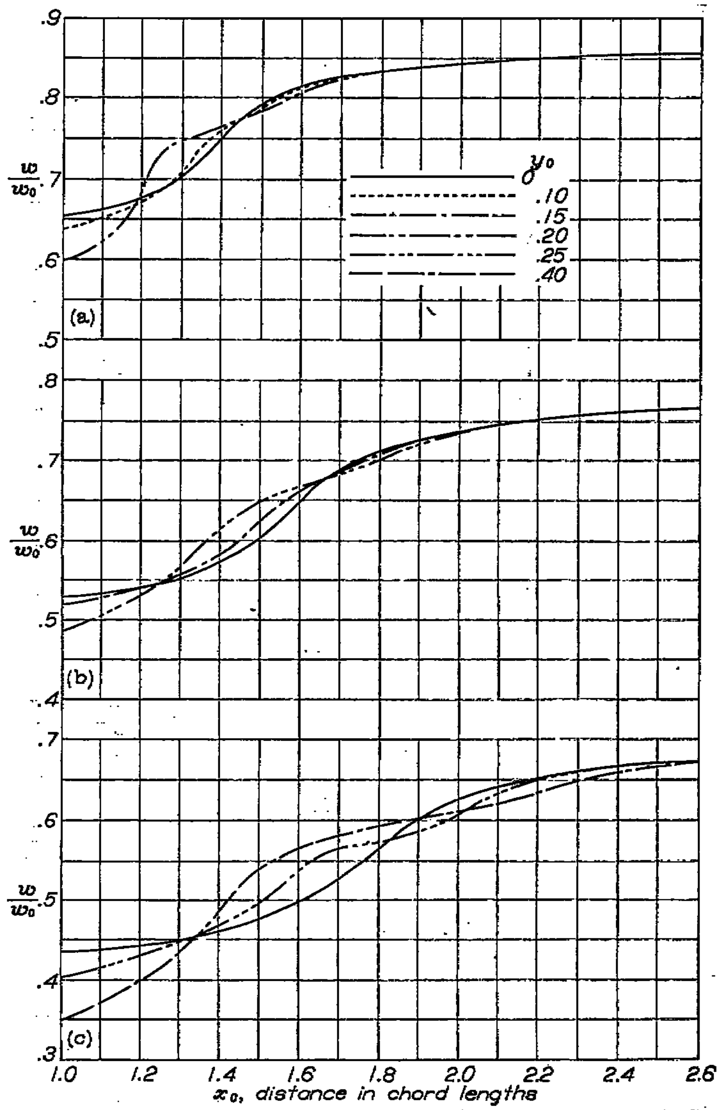
$$A_1=-\frac{1}{E_0\theta_0}\frac{z}{|z|} \quad (59)$$

The coefficient A_2 will not be evaluated since the first higher order term in z has been found. Thus, to the first order in y_0 and z_0, the downwash function w/w_0 is

$$\frac{w}{w_0}=\frac{2}{\pi E_0}\left(E_A+\int_0^{k_A}\frac{K_1-E_1}{k_1+\theta_0}\,dk_1\right)-\frac{z_0}{E_0\theta_0}\frac{z}{|z|} \quad (60)$$

DISCUSSION

The variable w/w_0 (i. e., $(w_P+w_W)/w_0$) represents the total downwash behind the wing divided by the induced vertical velocity on the wing itself. If ϵ is the downwash angle and α the angle of attack of the wing, then $w/w_0=d\epsilon/d\alpha$.



(a) $\theta_0=0.4$.
(b) $\theta_0=0.6$.
(c) $\theta_0=0.8$.

FIGURE 8.—Variation of the downwash in the x_0y_0 plane downstream from the trailing edge for various span stations.

In figure 8, the downwash in the xy plane is presented for various θ_0's and spanwise stations and for all values of x from the trailing edge to a point where the asymptotic value is closely approached. The region covered in the y direction extends from the x axis out to about (1/2) θ_0, where in the coordinate system used x_0 equals x/c_0, y_0 equals $\beta y/c_0$, and θ_0 is the semispan of the wing. Figure 8 can be used to assess the accuracy of the approximation, given by equation (60), that the value of downwash is independent of y in the neighborhood of the x axis. Within about a half span from the trailing edge of the wing no general statement can be made as to the variation of w/w_0 in the y direction. For distances greater than a half span from the trailing edge, however, the variation is quite uniform and w/w_0 deviates from its value at $y=0$ only slightly for $-\frac{1}{2}\theta_0<y_0<\frac{1}{2}\theta_0$.

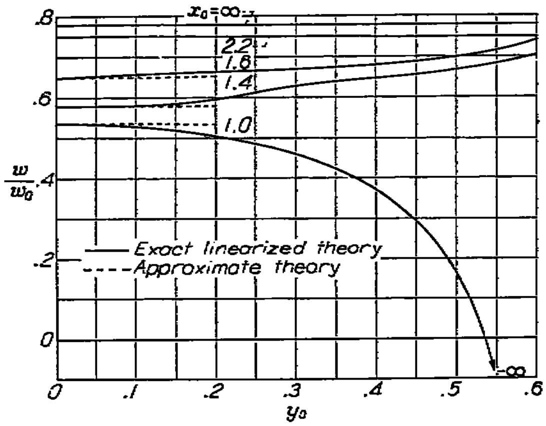
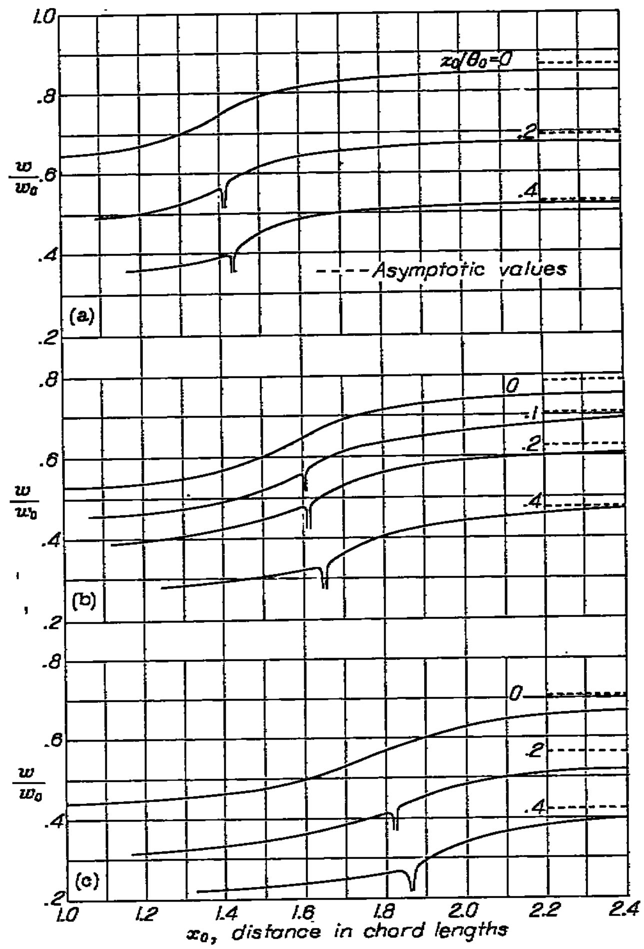


FIGURE 9.—Variation of downwash across the span at various stations downstream of wing trailing edge for $\theta_0=0.6$.

For $\theta_0=0.6$, a more extensive study was made of the variation of downwash with y. Figure 9 represents values of w/w_0 across the span for several positions behind the trailing edge. Immediately behind the trailing edge the value of w/w_0 falls and approaches $-\infty$ as the wing tip is approached. However, at 0.4 of a root chord behind the trailing edge ($x_0=1.4$), w/w_0 rises and reaches the value of 0.7 as the wing tip is reached. At $x_0=2.2$, the spanwise variation of w/w_0 is essentially constant. Although equation (60) is applicable only for region A, it is seen from figure 9 that the approximation that the downwash does not vary with y is useful out to about a third of a semispan for all values of x.

The variation of downwash in the xz plane is presented in figure 10. The curves represent values of w/w_0 from the trailing-edge wave downstream to a point where the asymptotic value is closely approached. In the immediate vicinity of the Mach cones from the trailing-edge tips (i. e., $x_0\approx 1+\theta_0$) the curves were not continued because w/w_0 becomes very large and approaches negative infinity as the Mach cone is approached. Since this effect results from infinitely large values of the radial component of induced velocity at the Mach cone, it does not exist in the $z_0=0$ plane. Such a behavior is consistent with the mathematical idealization of infinite pressures at the leading edge and of an abrupt fall of load at the trailing edge. However, in an actual flow field where these phenomena do not exist the flow will experience a milder change in passing across the Mach cone. Even in the theoretical results presented in this report the growth of the vertical induced velocity in the neighborhood of the Mach cone is logarithmic, and the interval in which w/w_0 is appreciably distorted from the general trend is very small.



(a) $\theta_0=0.4$.
(b) $\theta_0=0.6$.
(c) $\theta_0=0.8$.

FIGURE 10.—Variation of the downwash in the x_0z_0 plane downstream from the trailing edge.

Some further insight into the behavior of w in the vicinity of the Mach cone from the trailing-edge tips can be obtained by studying a single vortex which extends infinitely far ahead of the origin at an oblique angle to the flow and infinitely far behind the origin parallel to the flow (fig. 11). The half of the vortex which extends ahead makes an angle with the free-stream direction less than the Mach angle so that the

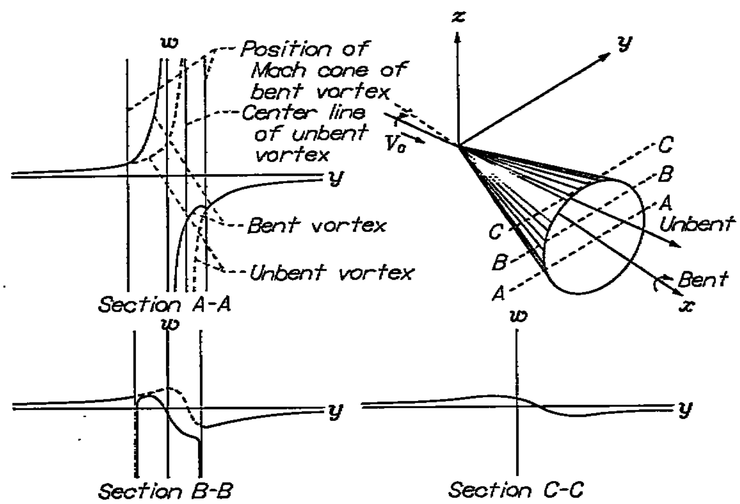
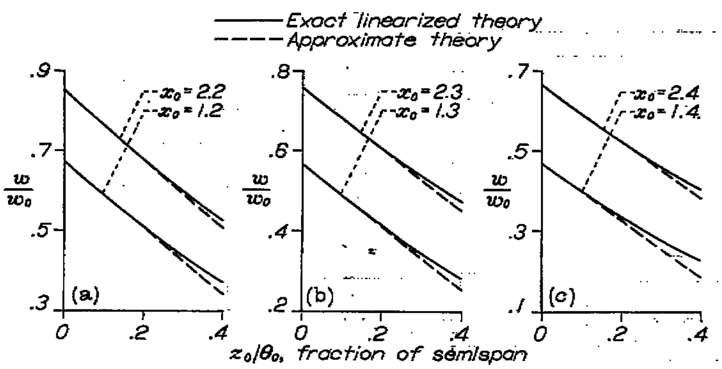


FIGURE 11.—Induced vertical velocity field for bent and unbent supersonic oblique vortex making an angle with the free stream less than the Mach angle.

component of free-stream velocity normal to it will be subsonic. Thus, outside of the Mach cone originating at the sudden bend in the vortex at the origin, the flow will be exactly like that of a linearized compressible subsonic vortex with a superimposed uniform velocity parallel to the line of the vortex. Inside the Mach cone, however, the flow is completely changed. Figure 11 gives an indication of the change. The term "bent" vortex refers to the vortex along the x axis which is turned suddenly at the origin from the angle it had maintained from $-\infty$. The term "unbent" vortex on the other hand refers to a vortex which maintains the same angle from $-\infty$ to $+\infty$. The unbent vortex is included in figure 11 for comparative purposes. The figure shows that on the $z=0$ plane (section AA) the downwash is finite and continuous in passing through the Mach cone, but that above the $z=0$ plane (section BB) the value of w becomes infinite as the cone surface is approached from the inside. This behavior at the Mach cone may aid in interpreting the discontinuity in the results for the complete wing as given in figure 10.



(a) $\theta_0=0.4$. (b) $\theta_0=0.6$. (c) $\theta_0=0.8$.

FIGURE 12.—Variation of downwash in the x_0z_0 plane at various positions on x_0 axis.

Approximate values of downwash in the xz plane computed from equation (60) are compared in figure 12 with the results from the exact solution. The approximation that to a first order the variation of w/w_0 with z_0/θ_0 is linear with a slope $-1/E_0$ is seen to be useful up to about a third of a semispan.

Values of w/w_0 were not computed for points off the xz and xy planes; however, the methods given in the report are general and directly applicable. The results already given would indicate that the approximate solution is valid in the vicinity of $(1/3)\theta_0$ about the x axis. This assumption can be checked for large distances behind the trailing edge by considering the flow field as x_0 approaches ∞. Thus figure 13 shows a comparison between the exact value of w/w_0 derived by means of the linearized equation and the approximate method based upon the use of a generalized Taylor's expansion. The agreement is seen to be satisfactory out to about one-third of a semispan either vertically or horizontally from the x axis.

Throughout the analysis it was obvious that the calculation of the downwash due to the doublets on the wake w_W was much simpler to perform than the calculation of the downwash due to the doublets on the plan form w_P. For example, the formulas for the downwash on the x axis were

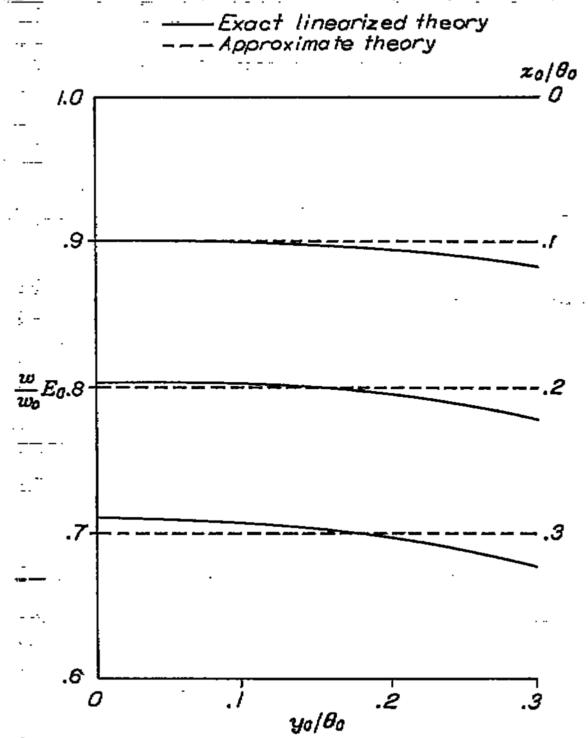


FIGURE 13.—Downwash at a large distance behind triangular wing.

given in terms of the elliptic integrals E and K; for w_W the evaluation of E and K was sufficient but for w_P a numerical integration involving E and K was necessary. Therefore, in calculating the downwash for wings with plan forms other

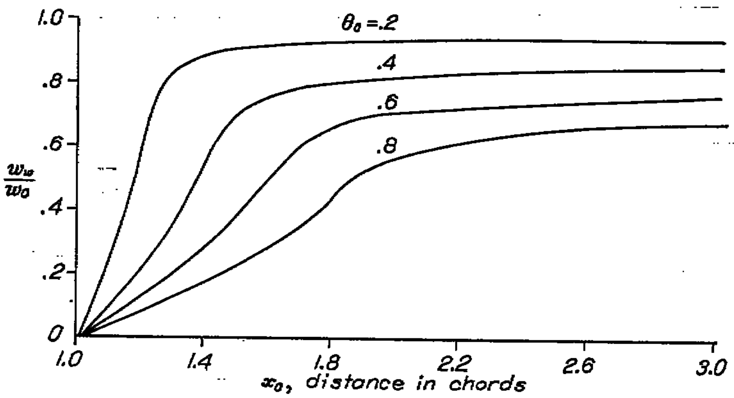


FIGURE 14.—Variation of the part of downwash on x axis induced by doublets in wake with distance downstream in chord lengths, $x_0=x/c_0$. Triangular wing.

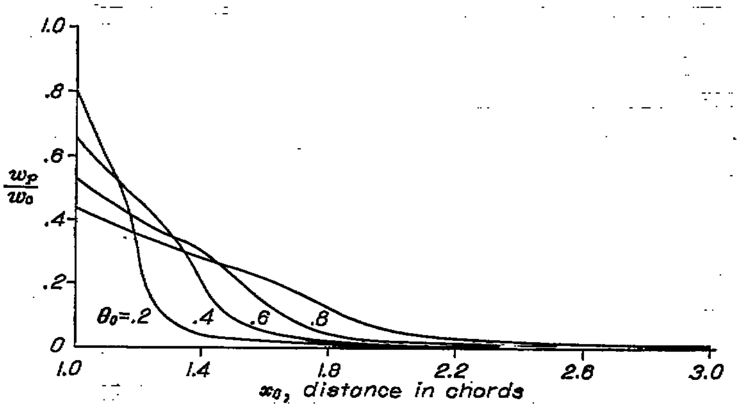


FIGURE 15.—Variation of the part of downwash on x axis induced by doublets on the plan form with distance downstream in chord lengths, $x_0=x/c_0$. Triangular wing.

than triangular, it is useful to know in what regions the contribution of w_P to w is small. For this purpose a comparison of w_W with w_P along the x axis of the triangular wing is shown in figures 14 and 15. Figure 14 gives the value w_W/w_0, figure 15 the value of w_P/w_0, and figure 16 the total downwash $(w_W+w_P)/w_0$ or just w/w_0. An inspection of figure 15 shows

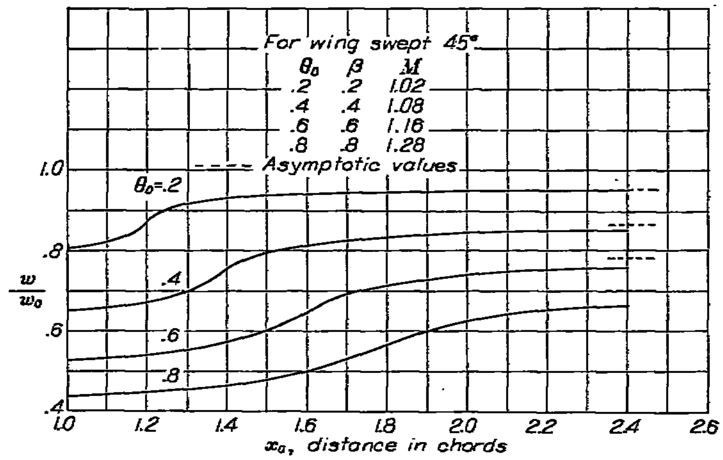


FIGURE 16.—Variation of the total downwash on x axis behind a triangular wing swept behind Mach cone with distance downstream in chord lengths, $x_0=x/c_0$.

that the effect of the doublets on the plan form dies out rapidly behind the point $x_0=1+\theta_0$, that is behind the point of

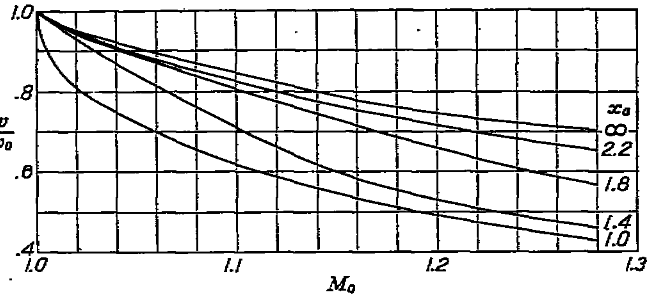


FIGURE 17.—Variation of downwash on x axis with Mach number at various positions downstream of trailing edge. $\psi=45°$.

intersection of the x axis with the Mach cone from the trailing-edge tip.

An indication of the variation of downwash with Mach number is given in figure 17. This figure shows values of w/w_0 on the x axis plotted as a function of M_0 for various values of x_0. The value of the sweepback angle is 45°, and the Mach number range covered could be extended to 1.4 and the leading edge would remain subsonic.

AMES AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
MOFFETT FIELD, CALIF., *Nov. 9, 1948.*

APPENDIX A

EVALUATION OF SPECIAL INTEGRALS

Integral I_1

Since there are no singularities in I_1, the finite part sign may be discarded. The linear term in the radical is eliminated by the transformation $\eta=(\gamma_1+\delta_1 t)/(1+t)$ and the integral becomes

$$I_1=\sqrt{\frac{(\mu'-\gamma_1)(\gamma_1-\mu)}{(\delta_1-\gamma_1)\gamma_1^3}}\int_{\frac{\mu'-\gamma_1}{\delta_1-\mu'}}^{\frac{\mu-\gamma_1}{\delta_1-\mu}}\frac{\sqrt{1-[(\delta_1-\mu')/(\mu'-\gamma_1)]^2t^2}}{[1-(\delta_1/\gamma_1)^2t^2]^{3/2}}\,dt \tag{A1}$$

The expressions for δ_1 and γ_1 may be combined to give the useful identities

$$\xi^2=\gamma_1\delta_1$$

and

$$(\gamma_1-\mu)(\delta_1-\mu')=(\mu'-\gamma_1)(\delta_1-\mu)$$

Noting that the integrand is an even function, equation (A1) may be reduced to the canonical form

$$I_1=2\sqrt{\frac{(\mu'-\gamma_1)(\gamma_1-\mu)}{(\delta_1-\gamma_1)\gamma_1^3}}\left(\frac{\mu'-\gamma_1}{\delta_1-\mu'}\right)\int_0^1\frac{\sqrt{1-\omega^2}}{(1-k_1^2\omega^2)^{3/2}}\,d\omega \tag{A2}$$

by the substitution

$$\omega=\frac{\delta_1-\mu'}{\mu'-\gamma_1}\,t$$

By the introduction of the Jacobian elliptic functions (reference 10) in the transformation $\omega=sn\,u$, the integration may be completed, and

$$I_1=2\sqrt{\frac{(\mu'-\gamma_1)(\gamma_1-\mu)}{(\delta_1-\gamma_1)\gamma_1^3}}\left(\frac{\mu'-\gamma_1}{\delta_1-\mu'}\right)\int_0^{K_1}cd^2\,u\,du=$$

$$\frac{1}{\xi^2}\sqrt{\frac{2\xi(\mu'-\mu)}{k_1}}\,(K_1-E_1) \tag{A3}$$

where $cd\,u\equiv cn\,u/dn\,u$.

Integral I_2

As the first step in reducing I_2 to canonical form the integral is written

$$I_2=\left|\overline{\int_\mu^\xi\left[1+\frac{\xi^2-\xi(\mu+\mu')+\mu\mu'}{2\xi(\eta-\xi)}-\frac{\xi^2+\xi(\mu+\mu')+\mu\mu'}{2\xi(\eta+\xi)}\right]\frac{d\eta}{\sqrt{(\xi+\eta)(\mu'-\eta)}\sqrt{(\xi-\eta)(\eta-\mu)}}}\right. \tag{A4}$$

In this case the following identities may be obtained directly from the definitions of γ_2 and δ_2:

$$(\gamma_2-\mu)(\xi-\delta_2)+(\gamma_2-\xi)(\mu-\delta_2)=0$$

and

$$(\xi+\gamma_2)(\mu'-\delta_2)+(\mu'-\gamma_2)(\xi+\delta_2)=0$$

The transformations $\eta=\dfrac{\gamma_2+\delta_2 t}{1+t}$ and $\omega=\dfrac{\delta_2-\mu}{\gamma_2-\mu}\,t$ are made and after algebraic simplifications equation (A4) becomes

$$I_2=\frac{\delta_2-\gamma_2}{(\mu'-\gamma_2)(\delta_2-\mu)}\sqrt{\frac{\mu'-\mu}{2\xi}}\left\{\left[1-\frac{(\mu'-\gamma_2)(\mu+\xi)(\mu'+\xi)}{(\delta_2-\mu')(\delta_2+\xi)(2\xi)}-\frac{(\gamma_2-\mu)(\xi-\mu)(\xi-\mu')}{(\delta_2-\mu)(\gamma_2-\xi)(2\xi)}\right]\left[\int_{-1}^1\frac{d\omega}{\sqrt{(1-k_2^2\omega^2)(1-\omega^2)}}\right]-\right.$$

$$\left.\left[\frac{(\mu+\xi)(\mu'+\xi)}{2\xi(\gamma_2+\xi)}\left(1-\frac{\mu'-\gamma_2}{\delta_2-\mu'}\right)\right]\int_{-1}^1\frac{1+k_2\omega}{(1-k_2^2\omega^2)^{3/2}\sqrt{1-\omega^2}}\,d\omega+\frac{(\xi-\mu)(\xi-\mu')}{2\xi(\gamma_2-\xi)}\left(1+\frac{\gamma_2-\mu}{\delta_2-\mu}\right)\left|\overline{\int_{-1}^1\frac{1+\omega}{(1-\omega^2)^{3/2}\sqrt{1-k_2^2\omega^2}}\,d\omega}\right.\right\} \tag{A5}$$

By applying the fundamental properties of even and odd functions, the first two integrals in equation (A5) can readily be integrated. The procedure for handling the finite part sign over the third integral will be considered in detail. Since,

$$\left|\overline{\int_{-1}^1\frac{1+\omega}{(1-\omega^2)\sqrt{(1-k_2^2\omega^2)(1-\omega^2)}}\,d\omega}\right.=2\left|\overline{\int^1\frac{d\omega}{(1-\omega^2)^{3/2}\sqrt{1-k_2^2\omega^2}}}\right.$$

and if

$$f(\omega)=\frac{1}{(1+\omega)^{3/2}\sqrt{1-k_2^2\omega^2}}$$

and

$$f(1)=\frac{1}{2^{3/2}\sqrt{1-k_2^2}}$$

then by equation (6)

$$2\int_0^1\frac{d\omega}{(1-\omega^2)^{3/2}\sqrt{1-k_2^2\omega^2}}=2\left[\int_0^1\frac{d\omega}{(1-\omega^2)^{3/2}\sqrt{1-k_2^2\omega^2}}-\int_0^1\frac{d\omega}{2^{3/2}(1-\omega)^{3/2}\sqrt{1-k_2^2}}-\frac{1}{\sqrt{2(1-k_2^2)}}\right]=$$

$$2\left[\int_0^1\frac{d\omega}{(1-\omega^2)^{3/2}\sqrt{1-k_2^2\omega^2}}-\lim_{\omega\to1}\frac{1}{\sqrt{2(1-\omega)(1-k_2^2)}}\right]=$$

$$2\left[\int_0^{K_2}\frac{du}{cn^2u}-\lim_{\omega\to1}\frac{1}{\sqrt{2(1-\omega)(1-k_2^2)}}\right]=$$

$$2\left\{\left[K_2+\lim_{\omega\to1}\frac{\sqrt{1-k_2^2\omega^2}}{(1-k_2^2)\sqrt{1-\omega^2}}\,\omega-\frac{E_2}{1-k_2^2}\right]-\lim_{\omega\to1}\frac{1}{\sqrt{2(1-\omega)(1-k_2^2)}}\right\}=$$

$$2\left(K_2-\frac{E_2}{1-k_2^2}\right)$$

The solution of equation (A5) becomes, after algebraic simplification

$$I_2=2\left\{\sqrt{\frac{k_2}{(\xi-\mu)(\xi-\mu')}}\left[\frac{\delta_2(\mu-\mu'+2\xi)-2\mu\xi}{\xi^2}\right]K_2-\frac{1}{2\xi^2}\sqrt{\frac{(\xi-\mu)(\xi+\mu')}{k_2}}\,E_2\right\} \tag{A6}$$

Integral I_3

The procedure for integrating I_3 is similar to that previously discussed in connection with I_1. In this case, the integral is canonicalized by the substitution $\omega=\sqrt{\frac{\delta_3}{\gamma_3}}\,t$ and the solution may be written

$$I_3=\frac{1}{\xi^2}\sqrt{\frac{2\xi(\mu'-\mu)}{k_3}}\,(K_3-E_3) \tag{A7}$$

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